Numerical Analysis

By,

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III YEAR - VI SEMESTER
COURSE CODE: 7BMAE2A

ELECTIVE COURSE - II (A) – NUMERICAL ANALYSIS

Unit – I

Solution of Algebraic and Transcendental equations – Introduction, Bisection Method, Iteration Method, Method of False position, Newton Raphson Method.

Unit – II

Interpolation : Finite differences – Forward differences, Backward differences, Central differences, Symbolic relations, Newton's formula for Interpolation – Interpolation with unevenly spaced points – Lagrange's Interpolation formula.

Unit – III

Numerical Differentiation and Integration – Introduction, Numerical Differentiation – Errors in Numerical Differentiation – Cubic Spline method – maximum and minimum values of a tabulated function, Numerical Integration – Trapezoidal Rule and Simpson's 1/3 and 3/8 rules.

Unit – IV

Matrices and Linear system of Equations – Gaussian Elimination method, Gauss – Jordan method, Modification of the Gauss method to compute the inverse – Method of Factorization – Iterative method – Jacobi and Gauss Seidal methods.

Unit – V

Numerical Solutions of Ordinary Differential Equations – Solution by Taylor Series, Picard's method of Successive approximations, Euler method, Modified Euler method Runge – Kutta Methods.

Text Book:

1. Introductory Methods of Numerical Analysis, (4^{th} Edition) by S.S.Sastry, PHI Learning Pvt. Ltd., New Delhi, 2009.

Unit I	Chapter 2sections 2.1 to 2.5
Unit II	Chapter 3 sections 3.3, 3.6, 3.9, 3.9.1.
Unit III	Chapter 5 sections 5.1, 5.2 - 5.2.2, 5.3, 5.4 – 5.4.1, 5.4.2, 5.4.3.
Unit IV	Chapter 6 sections 6.3.2, 6.3.3, 6.3.4, 6.4.
Unit V	Chapter 7 sections 7.2 to 7.4, 7.4.2, 7.5

Books for Reference:

1. Numerical Methods by P.Kandasamy and Others S.Chand Publications.
2. ... se,

Unit 1

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ELECTIVE COURSE - II (A) – NUMERICAL ANALYSIS

Unit – I

Solution of Algebraic and Transcendental equations – Introduction, Bisection Method, Iteration Method, Method of False position, Newton Raphson Method.

Unit - I

Following method for obtaining approximately soln for algebraic and transcendental equation.

i) Iteration method (successive approximation method)

ii) Aitken's Δ^2 method

iii) Bisection method (Bolzano method)

iv) Regula-falsi method (Method of false position)

v) Newton-Raphson method (Method of tangents)

To locate the root of an eqn $f(x)=0$ we use the following well known theorem in calculus.

If $f(x)$ is continuous in the interval $[a,b]$ and if $f(a)$ and $f(b)$ are of opposite signs then the eqn $f(x)=0$ has atleast one root lying between a and b.

I. Iteration Method

Pbm: 1

Use the method of iteration to find the real lying between 1 and 2 of the eqn $x^3-3x+1=0$.

Soln:

Let $f(x)=x^3-3x+1$

$f(1)=1^3-3(1)+1=-1$

$f(2)=2^3-3(2)+1$

$f(2)=8-6+1=3$

one root of $f(x) = 0$ lies between 1 and 2.

Now, $f(x) = 0$

$$x^3 - 3x + 1 = 0$$

$$x^3 = 3x - 1$$

$$x = (3x-1)^{1/3}$$

$$x = \phi(x)$$

$$\phi(x) = (3x-1)^{1/3}$$

$$\phi'(x) = \frac{1}{3}(3x-1)^{1-1/3} \cdot (3)$$

$$= (3x-1)^{-2/3}$$

$$\phi'(x) = \frac{1}{(3x-1)^{2/3}}$$

$\phi'(x) \Rightarrow x=1$ $\Rightarrow 0.63$
$\phi'(x) \Rightarrow x=2$ $\Rightarrow 0.342$

clearly $|\phi'(x)| < 1$ for all $x \in (1, 2)$.

If we take $x_0 = 2$, then $x_0, x_1, \cdots, x_n, \cdots$ is convergent.

$$x_1 = \phi(x_0) = \phi(2)$$

$$= (3(2)-1)^{1/3}$$

$$= (6-1)^{1/3}$$

$$= 5^{1/3}$$

$$x_1 = 1.7100$$

$$x_2 = \phi(x_1) = \phi(1.7100)$$

$$= (3(1.7100) - 1)^{1/3}$$

$$= (5.13 - 1)^{1/3}$$

$$= (4.13)^{1/3}$$

$$x_2 = 1.6044$$

$x_3 = \phi(x_2) = \phi(1.6044)$

$= (3(1.6044)-1)^{1/3}$

$= (4.8132-1)^{1/3}$

$= (3.8132)^{1/3}$

$x_3 = 1.5623$

$x_4 = \phi(x_3) = \phi(1.5623)$

$= (3(1.5623)-1)^{1/3}$

$= (4.6869-1)^{1/3}$

$= (3.6869)^{1/3}$

$x_4 = 1.5449$

$x_5 = \phi(x_4) = \phi(1.5449)$

$= (3(1.5449)-1)^{1/3}$

$= (4.6347-1)^{1/3}$

$= (3.6347)^{1/3}$

$x_5 = 1.5375$

$x_6 = \phi(x_5) = \phi(1.5375)$

$= (3(1.5375)-1)^{1/3}$

$= (4.6125-1)^{1/3}$

$= (3.6125)^{1/3}$

$x_6 = 1.5344$

$x_7 = \phi(x_6) = \phi(1.5344)$

$= (3(1.5344)-1)^{1/3}$

$= (4.6032-1)^{1/3}$

$= (3.6032)^{1/3}$

$x_7 = 1.5331$

$x_8 = \phi(x_7) = \phi(1.5331)$

$= (3(1.5331)-1)^{1/3}$

$= (4.5993-1)^{1/3}$

$= (3.5993)^{1/3}$

$x_8 = 1.5325$

$x_9 = \phi(x_8) = \phi(1.5325)$

$= (3(1.5325)-1)^{1/3}$

$= (4.5975-1)^{1/3}$

$= (3.5975)^{1/3}$

$x_9 = 1.5323$

$x_{10} = \phi(x_9) = \phi(1.5323)$

$= (3(1.5323)-1)^{1/3}$

$= (4.5969-1)^{1/3}$

$= (3.5969)^{1/3}$

$x_{10} = 1.5322$

Here, $x_9 = x_{10}$ correct to three places of decimals.

Hence an approximate value of the required root is 1.532.

Pbm : 2

Use the method of iteration to solve the eqn $3x - \log_{10} x = 6$.

Soln:

Let $f(x) = 3x - \log_{10} x - 6$

$f(0) = 0 - \log 0 - 6$

$= -1 - 6$

$f(0) = -7$

$f(1) = 3 - 0 - 6$

$f(1) = -3$

$f(2) = 3(2) - \log_{10}(2) - 6$

$= 6 - 0.3010 - 6$

$f(2) = -0.3010$

$f(3) = 3(3) - \log_{10}(3) - 6$

$= 9 - 0.4771 - 6$

$= 3 - 0.4771$

$f(3) = 2.5229$

$\therefore$ One root of $f(x) = 0$ lies between 2 and 3.

Now, $f(x) = 0$

$$3x - \log_{10} x - 6 = 0$$

$$3x = \log_{10} x + 6$$

$$x = \frac{1}{3}(\log_{10} x + 6)$$

$$x = \phi(x)$$

$$\phi(x) = \frac{1}{3}(\log_{10} x + 6)$$

$$(\therefore \log_{10} x = \log_e x \times \log_{10} e)$$

$$\phi(x) = \frac{1}{3}(\log_e x \times \log_{10} e + 6)$$

$$\phi'(x) = \frac{1}{3}\left(\frac{1}{x} \times \log_{10} e + 0\right)$$

$$\phi'(x) = \frac{1}{3}\left(\frac{\log_{10} e}{x}\right)$$

$$(\because \log_{10} e = 0.4343)$$

$\left(\frac{0.4343}{3 \times 2} = \frac{0.1447}{2} = 0.073\right)$

$\left(\frac{0.4343}{3 \times 3} = \frac{0.145}{3} = 0.05\right)$

$$|\phi'(x)| = \frac{0.4343}{3}\left|\frac{1}{x}\right|$$

$\therefore |\phi'(x)| < 1$ for all $\in (2, 3)$

If we take $x_0 = 2$, then $x_0, x_1, \dots, x_n$ is convergent.

$$x_1 = \phi(x_0) = \frac{1}{3}(\log_{10} 2 + 6)$$

$$= \frac{1}{3}(0.3010 + 6)$$

$$= \frac{6.3010}{3}$$

$$x_1 = 2.1003$$

$$x_2 = \phi(x_1) = \frac{1}{3}(\log_{10}(2.1003) + 6)$$

$= \frac{1}{3}(0.3223+6)$

$= \frac{6.3223}{3}$

$x_2 = 2.1074$

$x_3 = \phi(x_2) = \frac{1}{3}(\log_{10}(2.1074)+6)$

$= \frac{1}{3}(0.3238+6)$

$= \frac{6.3238}{3}$

$x_3 = 2.1079$

$x_4 = \phi(x_3) = \frac{1}{3}(\log_{10}(2.1079)+6)$

$= \frac{1}{3}(0.3239+6)$

$= \frac{6.3239}{3}$

$= 2.10796$

$x_4 = 2.1080$

$x_5 = \phi(x_4) = \frac{1}{3}(\log_{10}(2.1080)+6)$

$= \frac{1}{3}(0.3239+6)$

$= \frac{6.3239}{3}$

$x_5 = 2.1080$

Here, $x_4 = x_5$

Hence, an approximate value of the required root is 2.108.

Pbm : 3

Find real root of the eqn $\cos x = 3x-1$ correct to four decimal places using successive approximation method.

Soln: Let $f(x) = 3x - 1 - \cos x$.

$$f(0) = 0 - 1 - 1 = -2$$

$$f(\pi/2) = 3\pi/2 - 1 - 0$$

$$= \cancel{\frac{3\times 3.14}{2}} - 1$$

$$= \cancel{\frac{9.42}{2}} - 1$$

$$= \frac{3}{2} \times \frac{22}{7} - 1$$

$$= \frac{33}{7} - 1$$

$$= 4.7143 - 1$$

$$= 3.7143$$

$\therefore$ one root of the $f(x) = 0$ lies between 0 and $\pi/2$.

Now, $f(x) = 0$

$$3x - 1 - \cos x = 0$$

$$3x = 1 + \cos x$$

$$x = \frac{1}{3}(1 + \cos x)$$

$$x = \phi(x)$$

$$\phi(x) = \frac{1}{3}(1 + \cos x)$$

$$\phi'(x) = \frac{-\sin x}{3}$$

$\therefore |\phi'(x)| = \left|\frac{\sin x}{3}\right| < 1$ for all x in

$(0, \pi/2)$.

If we take $x_0 = 0$, then $x_0, x_1, \ldots, x_n$ is convergent.

$$x_1 = \phi(x_0) = \frac{1}{3}(1+\cos 0)$$
$$= \frac{1}{3}(2)$$
$$x_1 = 0.6667$$
$$x_2 = \phi(x_1) = \frac{1}{3}(1+\cos(0.6667))$$
$$= \frac{1}{3}(1+0.7859)$$
$$= \frac{1.7859}{3}$$
$$x_2 = 0.5953$$
$$x_3 = \phi(x_2) = \frac{1}{3}(1+\cos(0.5953))$$
$$= \frac{1}{3}(1+0.8280)$$
$$= \frac{1.8280}{3}$$
$$x_3 = 0.6093$$
$$x_4 = \phi(x_3) = \frac{1}{3}(1+\cos(0.6093))$$
$$= \frac{1}{3}(1+0.8200)$$
$$= \frac{1.8200}{3}$$
$$x_4 = 0.6067$$
$$x_5 = \phi(x_4) = \frac{1}{3}(1+\cos(0.6067))$$
$$= \frac{1}{3}(1+0.8215)$$
$$= \frac{1.8215}{3}$$

$x_5 = 0.6072$

$x_6 = \phi(x_5) = \frac{1}{3}(1+\cos 0.6072)$

$= \frac{1}{3}(1+0.8212)$

$= \frac{1.8212}{3}$

$x_6 = 0.6071$

$x_7 = \phi(x_6) = \frac{1}{3}(1+\cos(0.6071))$

$= \frac{1}{3}(1+0.8214)$

$= \frac{1.8214}{3}$

$x_7 = 0.6071$

Here, $x_6 = x_7$

Hence, an approximate value of the required root is 0.6071.

Exercise:

Solve the following eqns using iteration method.

1) $x^3+x^2-1=0$

soln:

Given: $x^3+x^2-1=0$

$x^3+x^2=1$

$x^2(x+1)=1$

$x^2=1, \quad x+1=1$

$x=1, \quad x=0$

$x(x^2+1)=1$

$x=1, \quad x^2+1=1$

$x^2=0$

$x=0$

$x=1, \quad x=0$

soln:

Let $f(x) = x^3 + x^2 - 1$

$f(0) = 0 + 0 - 1 = -1$

$f(1) = 1 + 1 - 1 = 1$

One roots of $f(x) = 0$ lies between 0 and 1.

Now, $f(x) = 0$

$x^3 + x^2 - 1 = 0$

$x^3 + x^2 = 1$

$x^2(x+1) = 1$

$x^2 = \frac{1}{x+1}$

$x = \frac{1}{\sqrt{x+1}}$

$x = \phi(x)$

$\phi(x) = \frac{1}{\sqrt{x+1}}$

$\phi(x) = (x+1)^{-1/2}$

$\phi'(x) = -\frac{1}{2}(x+1)^{-1/2-1} = -\frac{1}{2}(x+1)^{-3/2}$

$\phi'(x) = -\frac{1}{2(x+1)^{3/2}}$

$|\phi'(x)| = \frac{1}{2(x+1)^{3/2}} < 1$ for all $x \in (0,1)$

If we take $x_0 = 0$.

Then the sequence of successive approximation $x_0, x_1, \ldots, x_n$ is convergent.

$x_1 = \phi(x_0) = \phi(0) = (0+1)^{-1/2} = (1)^{-1/2}$

$x_1 = 1$

$$x_2 = \phi(x_1) = \phi(1) = (1+1)^{-1/2}$$
$$= (2)^{-1/2}$$
$$x_2 = 0.7071$$

$$x_3 = \phi(x_2) = \phi(0.7071) = (0.7071+1)^{-1/2}$$
$$= (1.7071)^{-1/2}$$
$$x_3 = 0.7654$$

$$x_4 = \phi(x_3) = \phi(0.7654)$$
$$= (0.7654+1)^{-1/2}$$
$$= (1.7654)^{-1/2}$$
$$x_4 = 0.7526$$

$$x_5 = \phi(x_4) = \phi(0.7526)$$
$$= (0.7526+1)^{-1/2}$$
$$= (1.7526)^{-1/2}$$
$$x_5 = 0.7554$$

$$x_6 = \phi(x_5) = \phi(0.7554)$$
$$= (0.7554+1)^{-1/2}$$
$$= (1.7554)^{-1/2}$$
$$x_6 = 0.7548$$

$$x_7 = \phi(x_6) = \phi(0.7548)$$
$$= (0.7548+1)^{-1/2}$$
$$= (1.7548)^{-1/2}$$
$$x_7 = 0.7548$$

Here $x_6 = x_7$ correct to four places of decimals.

Hence an approximate value of the required root is 0.7548.

Pbm: 4

Can we find a real root of the equation $x^3+x^2-1=0$ in the interval $[0,1]$ by the method of iteration?

soln:

$$x^3+x^2-1=0$$

$$x^3+x^2=1$$

$$x^2(x+1)=1$$

$$x^2=\frac{1}{x+1}$$

$$x=\frac{1}{\sqrt{x+1}}$$

$$x=\phi(x)$$

$$\phi(x)=(x+1)^{-1/2}$$

$$\phi'(x)=-\frac{1}{2}(x+1)^{-1/2-1}$$

$$=-\frac{1}{2}(x+1)^{-3/2}$$

$$\phi'(x)=\frac{-1}{2(x+1)^{3/2}}$$

$$|\phi'(x)|=\frac{1}{2(x+1)^{3/2}}<1 \text{ for all } x\in[0,1].$$

Hence, if we take $x_0=0$, then the sequence of approximations $x_0, x_1, \ldots, x_n$ is convergent and an approximate value of the roots can be obtained by the method of iteration.

Pbm: 5

can we apply iteration method to find the root of the equation $2x = \cos x + 3$ in $(0, \pi/2)$?

soln:

$$2x = \cos x + 3$$

$$x = \frac{1}{2}(\cos x + 3)$$

$$x = \phi(x)$$

$$\phi(x) = \frac{1}{2}(\cos x + 3)$$

$$\phi'(x) = \frac{1}{2}(-\sin x + 0)$$

$$\phi'(x) = \frac{-\sin x}{2}$$

$$|\phi'(x)| = \left|\frac{\sin x}{2}\right| < 1 \quad \text{for all } x \in (0, \pi/2)$$

Hence if we take $x_0 = 0$. Then the sequence of approximations $x_0, x_1, \ldots, x_n \ldots$ is convergent and an approximate value of the roots can be obtained by the method of iteration.

Exercise.

2) $3x - \sqrt{1+\sin x} = 0$

Let $f(x) = 3x - \sqrt{1+\sin x}$

$$f(0) = 3(0) - \sqrt{1+\sin(0)}$$

$$= 0 - \sqrt{1+0}$$

$$f(0) = -1$$

$$f(\pi/2) = 3\pi/2 - \sqrt{1+\sin \pi/2}$$

$$= 3\pi/2 - \sqrt{1+1}$$

$= 3\pi/2 - \sqrt{2}$

$= \cancel{\frac{3}{2} \times \frac{22}{7} - 1.4142}$

$= \cancel{4.7143 - 1.4142}$

$= \frac{3}{2} \times \frac{22}{7} - 1.4142$

$= \cancel{\frac{9.4286}{2} - 1.4142}$

$= \frac{9.4248}{2} - 1.4142$

$= 4.7124 - 1.4142$

$f(\pi/2) = 3.2982$

one root of $f(x)=0$ lies between 0 and $\pi/2$.

Now, $f(x)=0$

$3x - \sqrt{1+\sin x} = 0$

$3x = \sqrt{1+\sin x}$

$x = \frac{1}{3}\sqrt{1+\sin x}$

$x = \phi(x)$

$\phi'(x) = \frac{1}{3} \cdot \frac{1}{2} (1+\sin x)^{1/2 - 1} \cdot \cos x$

$= \frac{1}{6} (1+\sin x)^{-1/2} \cdot \cos x$

$$\phi'(x) = \frac{\cos x}{6\sqrt{1+\sin x}}$$

$$|\phi'(x)| = \left| \frac{\cos x}{6\sqrt{1+\sin x}} \right| < 1 \quad \text{for all } x \in (0, \pi/2)$$

Hence if we take $x_0 = 0$

Then the sequence of successive approximation $x_0, x_1, \ldots, x_n$ is convergent.

$$x_1 = \phi(x_0) = \phi(0) = \frac{1}{3}\sqrt{1+\sin(0)}$$

$$= \frac{1}{3}\sqrt{1+0}$$

$$= \frac{1}{3}$$

$$x_1 = 0.3333$$

$$x_2 = \phi(x_1) = \phi(0.3333)$$

$$= \frac{1}{3}\sqrt{1+\sin(0.3333)}$$

$$= \frac{1}{3}\sqrt{1+0.3272}$$

$$= \frac{1}{3}\sqrt{1.3272}$$

$$= \frac{1.1520}{3}$$

$$x_2 = 0.384$$

$$x_3 = \phi(x_2) = \phi(0.384) = \frac{1}{3}\sqrt{1+\sin(0.384)}$$

$$= \frac{1}{3}\sqrt{1+0.3746}$$

$$= \frac{1}{3}\sqrt{1.3746}$$

$$= \frac{1.1724}{3}$$

$$x_3 = 0.3908$$

$$x_4 = \phi(x_3) = \phi(0.3908) = \frac{1}{3}\sqrt{1+\sin(0.3908)}$$

$$= \frac{1}{3}\sqrt{1+0.3809}$$

$$= \frac{1}{3}\sqrt{1.3809}$$

$$= \frac{1.1751}{3}$$

$$x_4 = 0.3917$$

$x_5 = \phi(x_4) = \phi(0.3917) = \frac{1}{3}\sqrt{1+\sin(0.3917)}$

$= \frac{1}{3}\sqrt{1+0.3818}$

$= \frac{1}{3}\sqrt{1.3818}$

$= \frac{1.1755}{3}$

$x_5 = 0.3918$

$x_6 = \phi(x_5) = \phi(0.3918) = \frac{1}{3}\sqrt{1+\sin(0.3918)}$

$= \frac{1}{3}\sqrt{1+0.3819}$

$= \frac{1}{3}\sqrt{1.3819}$

$= \frac{1.1755}{3}$

$x_6 = 0.3918$

Hence, $x_5 = x_6$ correct to four places of decimals.

Hence, an approximate value of the required root is 0.3918

3) $x^3+x^2-100=0$

Soln:

Let $f(x) = x^3+x^2-100$

$f(1) = 1+1-100 = -98$

$f(2) = 8+4-100 = -88$

$f(3) = 27+9-100 = -64$

$f(4) = 64+16-100 = -20$

$f(5) = 125+25-100 = 50$

One root of $f(x)=0$ lies between 4 and 5.

Now, $f(x) = 0$

$$x^3+x^2-100=0$$

$$x^3+x^2=100$$

$$x^2(x+1)=100$$

$$x^2=\frac{100}{x+1}$$

$$x=\frac{\sqrt{100}}{\sqrt{x+1}}$$

$$x=\frac{10}{\sqrt{x+1}}=\phi(x)$$

$$\phi'(x)=10\left(-\tfrac{1}{2}(x+1)^{-1/2}\right)$$

$$=\frac{-10}{2}\cdot\frac{1}{\sqrt{1+x}}$$

$$\phi'(x)=\frac{-5}{\sqrt{1+x}}$$

$$|\phi'(x)|=\left|\frac{5}{\sqrt{1+x}}\right|<1 \text{ for all } x\in(4,5)$$

If we take $x_0=4$, then the sequence of successive approximation $x_0, x_1, \ldots, x_n$ is convergent.

$$x_1=\phi(x_0)=\phi(4)=\frac{10}{\sqrt{4+1}}=\frac{10}{\sqrt{5}}=\frac{10}{2.2361}$$

$$x_1=4.4721$$

$$x_2=\phi(x_1)=\phi(4.4721)=\frac{10}{\sqrt{4.4721+1}}$$

$$=\frac{10}{\sqrt{5.4721}}=\frac{10}{2.3393}$$

$$x_2=4.2749$$

$$x_3=\phi(x_2)=\phi(4.2749)=\frac{10}{\sqrt{4.2749+1}}$$

$$=\frac{10}{\sqrt{5.2749}}=\frac{10}{2.2967}$$

$x_3 = 4.3541$

$x_4 = \phi(x_3) = \phi(4.3541) = \frac{10}{\sqrt{4.3541+1}}$

$= \frac{10}{\sqrt{5.3541}} = \frac{10}{2.3139}$

$x_4 = 4.3217$

$x_5 = \phi(x_4) = \phi(4.3217) = \frac{10}{\sqrt{4.3217+1}}$

$= \frac{10}{\sqrt{5.3217}} = \frac{10}{2.3069}$

$x_5 = 4.3348$

$x_6 = \phi(x_5) = \phi(4.3348) = \frac{10}{\sqrt{4.3348+1}}$

$= \frac{10}{\sqrt{5.3348}} = \frac{10}{2.3097}$

$x_6 = 4.3296$

$x_7 = \phi(x_6) = \phi(4.3296) = \frac{10}{\sqrt{4.3296+1}}$

$= \frac{10}{\sqrt{5.3296}} = \frac{10}{2.3086}$

$x_7 = 4.3316$

$x_8 = \phi(x_7) = \phi(4.3316) = \frac{10}{\sqrt{4.3316+1}}$

$= \frac{10}{\sqrt{5.3316}} = \frac{10}{2.3090}$

$x_8 = 4.3309$

$x_9 = \phi(x_8) = \phi(4.3309) = \frac{10}{\sqrt{4.3309+1}}$

$= \frac{10}{\sqrt{5.3309}} = \frac{10}{2.3089}$

$x_9 = 4.3311$

$$x_{10} = \phi(x_9) = \phi(4.3311) = \frac{10}{\sqrt{4.3311+1}}$$

$$= \frac{10}{\sqrt{5.3311}} = \frac{10}{2.3089}$$

$x_{10} = 4.3311$ [Here, $x_9 = x_{10}$]

Hence, the approximate value of the required root is 4.3311.

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4) $2x - \log_{10} x - 7 = 0$

Soln:

Let $f(x) = 2x - \log_{10} x - 7$

$f(1) = 2 - \log_{10}(1) - 7$

$= 2 - 0 - 7$

$f(1) = -5$

$f(2) = 4 - \log_{10}(2) - 7$

$= 4 - 0.3010 - 7$

$f(2) = -3.3010$

$f(3) = 2(3) - \log_{10}(3) - 7$

$= 6 - 0.4771 - 7$

$f(3) = -1.4771$

$f(4) = 2(4) - \log_{10}(4) - 7$

$= 8 - 0.6021 - 7$

$f(4) = 0.3979$

$\therefore$ one root of $f(x) = 0$ lies between 3 and 4.

Now, $f(x) = 0 \Rightarrow 2x - \log_{10} x - 7 = 0$

$2x = \log_{10} x + 7$

$x = \frac{1}{2}(\log_{10} x + 7) = \phi(x)$

$(\log_{10} x = \log_e x \times \log_{10} e)$

$\phi(x) = \frac{1}{2}(\log_e x \times \log_{10} e + 7)$

$\phi'(x) = \frac{1}{2}\left[\log_{10} e \left(\frac{1}{x}\right) + 0\right]$

$= \frac{\log_{10} e}{2x}$

$= \frac{0.4343}{2x}$ $(\log_{10} e = 0.4343)$

$\phi'(x) = \frac{1}{2} \cdot \frac{0.4343}{x}$

$|\phi'(x)| = \frac{0.4343}{2}\left|\frac{1}{x}\right| < 1$ for all x

$\in (3,4)$

If we take $x_0 = 3$, then the sequence of successive approximation $x_0, x_1, \ldots, x_n$ is convergent.

$x_1 = \phi(x_0) = \phi(3) = \frac{\log_{10}(3) + 7}{2} = \frac{0.4771+7}{2}$

$= \frac{7.4771}{2}$

$x_1 = 3.7386$

$x_2 = \phi(x_1) = \phi(3.7386) = \frac{\log_{10}(3.7386)+7}{2}$

$= \frac{0.5727+7}{2} = \frac{7.5727}{2}$

$x_2 = 3.7864$

$$x_3 = \phi(x_2) = \phi(3.7864) = \frac{\log_{10}(3.7864)+7}{2}$$

$$= \frac{0.5782+7}{2} = \frac{7.5782}{2}$$

$$x_3 = 3.7891$$

$$x_4 = \phi(x_3) = \phi(3.7891) = \frac{\log_{10}(3.7891)+7}{2}$$

$$= \frac{0.5785+7}{2} = \frac{7.5785}{2}$$

$$x_4 = 3.7893$$

$$x_5 = \phi(x_4) = \phi(3.7893) = \frac{\log_{10}(3.7893)+7}{2}$$

$$= \frac{0.5786+7}{2} = \frac{7.5786}{2}$$

$$x_5 = 3.7893$$

Here $x_4 = x_5$, Hence an approximate value of the required root is 3.7893.

---

5) $1+\sin x - 2x = 0$

Let $f(x) = 1+\sin x - 2x$

$f(0) = 1+\sin(0) - 2(0) = 1$

$f(\pi/2) = 1+\sin \pi/2 - 2\pi/2$

$= 1+1-3.1416$

$= 2-3.1416$

$f(\pi/2) = -1.1416$

$\therefore$ one root of $f(x)=0$ lies between 0 and $\pi/2$.

Now, $f(x) = 0 \Rightarrow 1 + \sin x - 2x = 0$

$$1 + \sin x = 2x$$

$$x = \frac{1+\sin x}{2}$$

$$x = \phi(x) = \frac{1}{2}(1+\sin x)$$

$$\phi'(x) = \frac{1}{2}(+\cos x)$$

$$\phi'(x) = \frac{+\cos x}{2}$$

$\therefore |\phi'(x)| = \left|\frac{\cos x}{2}\right| < 1$ for all x in $(0, \pi/2)$

If we take $x_0 = 0$, then the sequence of successive approximation $x_0, x_1, \dots, x_n$ is convergent.

$$x_1 = \phi(x_0) = \phi(0) = \frac{1}{2}(1+\sin(0)) = \frac{1}{2}(1)$$

$$x_1 = \frac{1}{2} = 0.5$$

$$x_2 = \phi(x_1) = \phi(0.5) = \frac{1}{2}(1+\sin(0.5))$$

$$= \frac{1}{2}(1+0.4794)$$

$$= \frac{1.4794}{2}$$

$$x_2 = 0.7397$$

$$x_3 = \phi(x_2) = \phi(0.7397) = \frac{1}{2}(1+\sin(0.7397)) = \frac{1}{2}(1+0.6741)$$

$$x_3 = \frac{1.6741}{2} = 0.8371$$

$$x_4 = \phi(x_3) = \cancel{\phi(0.8399)} = \frac{1}{2}(1+\sin 0.8699)$$

$$\cancel{= \frac{1.8699}{2} =}$$

$$x_4 = \phi(x_3) = \phi(0.8371) = \frac{1}{2}(1+\sin(0.8371))$$

$$= \frac{1}{2}(1+0.7427)$$

$$x_4 = \frac{1.7427}{2} = 0.8714$$

$$x_5 = \phi(x_4) = \phi(0.8714) = \frac{1}{2}(1+\sin(0.8714))$$

$$= \frac{1}{2}(1+0.7652) = \frac{1.7652}{2}$$

$$x_5 = 0.8826$$

$$x_6 = \phi(x_5) = \phi(0.8826) = \frac{1}{2}(1+\sin(0.8826))$$

$$= \frac{1}{2}(1+0.7724) = \frac{1.7724}{2}$$

$$x_6 = 0.8862$$

$$x_7 = \phi(x_6) = \phi(0.8862) = \frac{1}{2}(1+\sin(0.8862))$$

$$= \frac{1+0.7747}{2} = \frac{1.7747}{2}$$

$$x_7 = 0.8874$$

$$x_8 = \phi(x_7) = \phi(0.8874) = \frac{1}{2}(1+\sin(0.8874))$$

$$= \frac{1+0.7754}{2} = \frac{1.7754}{2}$$

$$x_8 = 0.8877$$

$$x_9 = \phi(x_8) = \phi(0.8877) = \frac{1}{2}(1+\sin(0.8877))$$

$$= \frac{1+0.7756}{2} = \frac{1.7756}{2}$$

$$x_9 = 0.8878$$

Here, $x_8 = x_9$, correct to ~~first~~ three decimal places.

Hence, an approximate value of the required root is 0.887.

6) $e^x - 3x = 0$

Let $f(x) = e^x - 3x$

$f(0) = e^0 - 3(0) = 1$

$f(1) = e^1 - 3(1) = 2.7183 - 3$

$f(1) = -0.2817$

$\therefore$ one root of $f(x) = 0$ lies between 0 and 1.

Now, $f(x) = 0$

$e^x - 3x = 0$

$-3x = -e^x$

$x = \frac{e^x}{3}$

$x = \phi(x) = \frac{e^x}{3}$

$\phi'(x) = \frac{e^x}{3}$

$\therefore |\phi'(x)| = \left|\frac{e^x}{3}\right| < 1$ for all x in $(0, 1)$.

If we take $x_0 = 0$, Then the sequence of successive approximation $x_0, x_1, \ldots, x_n$ is convergent.

$x_1 = \phi(x_0) = \phi(0) = \frac{e^0}{3} = \frac{1}{3} = 0.3333$

$$x_2 = \phi(x_1) = \phi(0.3333) = \frac{e^{0.3333}}{3}$$

$$x_2 = \frac{1.3957}{3} = 0.4652$$

$$x_3 = \phi(x_2) = \phi(0.4652) = \frac{e^{0.4652}}{3}$$

$$x_3 = \frac{1.5923}{3} = 0.5308$$

$$x_4 = \phi(x_3) = \phi(0.5308) = \frac{e^{0.5308}}{3}$$

$$x_4 = \frac{1.7003}{3} = 0.5668$$

$$x_5 = \phi(x_4) = \phi(0.5668) = \frac{e^{0.5668}}{3}$$

$$x_5 = \frac{1.7626}{3} = 0.5875$$

$$x_6 = \phi(x_5) = \phi(0.5875) = \frac{e^{0.5875}}{3}$$

$$x_6 = \frac{1.7995}{3} = 0.5998$$

$$x_7 = \phi(x_6) = \phi(0.5998) = \frac{e^{0.5998}}{3}$$

$$x_7 = \frac{1.8218}{3} = 0.6073$$

$$x_8 = \phi(x_7) = \phi(0.6073) = \frac{e^{0.6073}}{3}$$

$$x_8 = \frac{1.8355}{3} = 0.6118$$

$$x_9 = \phi(x_8) = \phi(0.6118) = \frac{e^{0.6118}}{3}$$

$$x_9 = \frac{1.8437}{3} = 0.6146$$

$$x_{10} = \phi(x_9) = \phi(0.6146) = \frac{e^{0.6146}}{3}$$

$$x_{10} = \frac{1.8489}{3} = 0.6163$$

$$x_{11} = \phi(x_{10}) = \phi(0.6163) = \frac{e^{0.6163}}{3}$$

$$x_{11} = \frac{1.8521}{3} = 0.6174.$$

$$x_{12} = \phi(x_{11}) = \phi(0.6174) = \frac{e^{0.6174}}{3}$$

$$= \frac{1.8541}{3} = 0.6180$$

$$x_{13} = \phi(x_{12}) = \phi(0.6180) = \frac{e^{0.6180}}{3}$$

$$x_{13} = \frac{1.8552}{3} = 0.6184$$

$$x_{14} = \phi(x_{13}) = \phi(0.6184) = \frac{e^{0.6184}}{3}$$

$$x_{14} = \frac{1.8560}{3} = 0.6187$$

$$x_{15} = \phi(x_{14}) = \phi(0.6187) = \frac{e^{0.6187}}{3}$$

$$x_{15} = \frac{1.8565}{3} = 0.6188$$

$$x_{16} = \phi(x_{15}) = \phi(0.6188) = \frac{e^{0.6188}}{3}$$

$$x_{16} = \frac{1.8567}{3} = 0.6189$$

$$x_{17} = \phi(x_{16}) = \phi(0.6189) = \frac{e^{0.6189}}{3}$$

$$x_{17} = \frac{1.8569}{3} = 0.6190$$

$$x_{18} = \phi(x_{17}) = \phi(0.6190) = \frac{e^{0.6190}}{3}$$

$$x_{18} = \frac{1.8571}{3} = 0.6190$$

Here $x_{17} = x_{18}$, correct to four decimal places. Hence, an approximate value of the required root is 0.6190.

8) Use the method of iteration to solve the eqn $3x+\sin x = e^x$.

Let $f(x) = 3x+\sin x - e^x$

$f(0) = 3(0)+\sin 0 - e^0 = -1$

$f(\pi/2) = 3\pi/2 + \sin\pi/2 - e^{\pi/2}$

$= 4.7124 + 1 - 4.8105$

$f(\pi/2) = 0.9019$

$\therefore$ one root of $f(x)=0$ lies between 0 and $\pi/2$.

Now, $f(x) = 0$

$3x + \sin x - e^x = 0$

$3x = e^x - \sin x$

$$x = \frac{e^x - \sin x}{3}$$

$$x = \phi(x) = \frac{e^x - \sin x}{3}$$

$\phi(x) = \frac{1}{3}(e^x - \sin x)$

$\phi'(x) = \frac{1}{3}(e^x - \cos x)$

$\therefore |\phi'(x)| = \left|\frac{1}{3}(e^x - \cos x)\right| < 1$ for all $x \in (0, \pi/2)$

If we take $x_0 = 0$, then the sequence of successive approximation $x_0, x_1, \ldots$

is convergent.

$x_1 = \phi(x_0) = \phi(0) = \frac{e^0 - \sin 0}{3} = \frac{1}{3}$

$x_1 = 0.3333$

$x_2 = \phi(x_1) = \phi(0.3333) = \frac{e^{0.3333} - \sin(0.3333)}{3}$

$= \frac{1.3957 - 0.3272}{3} = \frac{1.0685}{3}$

$x_2 = 0.3562$

$x_3 = \phi(x_2) = \phi(0.3562) = \frac{e^{0.3562} - \sin(0.3562)}{3}$

$= \frac{1.4279 - 0.3487}{3} = \frac{1.0792}{3}$

$x_3 = 0.3597$

$x_4 = \phi(x_3) = \phi(0.3597) = \frac{e^{0.3597} - \sin(0.3597)}{3}$

$= \frac{1.4329 - 0.3520}{3} = \frac{1.0809}{3}$

$x_4 = 0.3603$

$x_5 = \phi(x_4) = \phi(0.3603) = \frac{e^{0.3603} - \sin(0.3603)}{3}$

$= \frac{1.4338 - 0.3526}{3} = \frac{1.0812}{3}$

$x_5 = 0.3604$

$x_6 = \phi(x_5) = \phi(0.3604) = \frac{e^{0.3604} - \sin(0.3604)}{3}$

$= \frac{1.4339 - 0.3526}{3} = \frac{1.0813}{3}$

$x_6 = 0.3604$

Here $x_5 = x_6$, correct to four decimal places.

Hence, an approximate value of the required root is 0.3604.

II. AITKEN'S Δ^2 METHOD.

Pbm: 1

Use Aitken's Δ^2 method find the real root lying between 1 and 2 of the eqn $x^3 - 3x + 1 = 0$

soln:

Let $f(x) = x^3 - 3x + 1$

$f(1) = 1 - 3 + 1 = -1$

$f(2) = 8 - 6 + 1 = 3$

$\therefore$ one root of $f(x) = 0$ lies between 1 and 2.

Now, $f(x) = 0$

$$x^3 - 3x + 1 = 0$$

$$x^3 = 3x - 1$$

$$x = (3x-1)^{1/3}$$

$$x = \phi(x) = (3x-1)^{1/3}$$

$$\phi'(x) = \frac{1}{3}(3x-1)^{1/3 - 1}(3)$$

$$= (3x-1)^{-2/3}$$

$$\phi'(x) = \frac{1}{(3x-1)^{2/3}}$$

$\therefore |\phi'(x)| = \left| \dfrac{1}{(3x-1)^{2/3}} \right| < 1$ for all $x \in (1,2)$

If we take $x_0 = 2$, then, the sequence of successive first three approximation x_0, x_1, x_2, x_3 is convergent. (using iteration method)

$$\cancel{x_1 = \phi(x_0) = \phi(2) = 8-6+1 = 3}$$
$$\cancel{x_2 = \phi(x_1) = \phi(3) =}$$

$$x_1 = \phi(x_0) = \phi(2) = (3(2)-1)^{1/3}$$
$$= (5)^{1/3}$$
$$= 1.7099$$
$$x_1 = 1.7100$$

$$x_2 = \phi(x_1) = \phi(1.7100) = (3(1.7100)-1)^{1/3}$$
$$= (5.13-1)^{1/3}$$
$$= (4.13)^{1/3}$$
$$x_2 = 1.6044$$

$$x_3 = \phi(x_2) = \phi(1.6044) = (3(1.6044)-1)^{1/3}$$
$$= (4.8132-1)^{1/3}$$
$$= (3.8132)^{1/3}$$
$$x_3 = 1.5623$$

$\therefore$ The first three approximation are

$$x_1 = 1.7100, \; x_2 = 1.6044, \; x_3 = 1.5623$$

From the 1st three consective iterations,

$$\Delta x_2 = x_3 - x_2 = 1.5623 - 1.6044 = -0.0421$$

$$\Delta^2 x_1 = \Delta(\Delta x_1)$$
$$= \Delta(x_2 - x_1)$$
$$= \Delta x_2 - \Delta x_1$$
$$= x_3 - x_2 - (x_2 - x_1)$$
$$= x_3 - 2x_2 + x_1$$
$$= 1.5623 - 2(1.6044) + 1.7100$$
$$= 1.5623 - 3.2088 + 1.7100$$
$$\Delta^2 x_1 = 0.0635$$

By Aitken's iteration,

$$\alpha = x_{i+1} - \frac{(\Delta x_i)^2}{\Delta^2 x_{i-1}}$$
$$x_4 = x_{2+1} - \frac{(\Delta x_2)^2}{\Delta^2 x_{2-1}}$$
$$= x_3 - \frac{(\Delta x_2)^2}{\Delta^2 x_1}$$
$$= 1.5623 - \frac{(-0.0421)^2}{0.0635}$$
$$= 1.5623 - \frac{0.00177241}{0.0635}$$
$$= 1.5623 - 0.0279118$$
$$x_4 = 1.5343882$$

x_4 is correct to four decimal places as 1.5344.

$$x_4 = 1.5344.$$

Pbm: 2.

Find a real root of the eqn $\cos x = 3x-1$ correct to the decimal places using (i) Iteration method (ii) Aitken's method.

soln:

i) iteration method:

Let $f(x) = \cos x - 3x + 1$

$f(0) = \cos 0 - 3(0) + 1$

$= 1 - 0 + 1$

$f(0) = 2$

$f(\pi/2) = \cos \pi/2 - 3\pi/2 + 1$

$= 0 - \frac{9.4286}{2} + 1$

$= -4.7143 + 1$

$f(\pi/2) = -3.7143$

$\therefore$ one root of $f(x) = 0$ lies between 0 and $\pi/2$.

Now, $f(x) = 0$

$\cos x - 3x + 1 = 0$

$-3x = -(1+\cos x)$

$x = \frac{1+\cos x}{3}$

$x = \phi(x) = \frac{1}{3}(1+\cos x)$

$\phi'(x) = -\frac{\sin x}{3}$

$\therefore |\phi'(x)| = \left|\frac{\sin x}{3}\right| < 1$ for all $x \in (0, \pi/2)$

If we take $x_0 = 0$, then the sequence of successive approximations $x_0, x_1, \ldots, x_n$ is convergent.

$$x_1 = \phi(0) = \frac{1}{3}(1+\cos 0) = \frac{2}{3} = 0.6667$$

$$x_2 = \phi(0.6667) = \frac{1}{3}(1+\cos(0.6667)) = \frac{1.7859}{3}$$

$$x_2 = 0.5953$$

$$x_3 = \phi(0.5953) = \frac{1}{3}(1+\cos(0.5953)) = \frac{1.8280}{3}$$

$$x_3 = 0.6093$$

$$x_4 = \phi(0.6093) = \frac{1}{3}(1+\cos(0.6093)) = \frac{1.8200}{3}$$

$$x_4 = 0.6067$$

$$x_5 = \phi(0.6067) = \frac{1}{3}(1+\cos(0.6067)) = \frac{1.8215}{3}$$

$$x_5 = 0.6072$$

$$x_6 = \phi(0.6072) = \frac{1}{3}(1+\cos(0.6072)) = \frac{1.8212}{3}$$

$$x_6 = 0.6071$$

Here $x_5 = x_6$, correct to three decimal places.

Hence, an approximate value of the required root is 0.607

ii) Aitken's method:

From (i) we have the first three iteration to be calculated.

$x_1 = 0.6667$

$x_2 = 0.5953$

$x_3 = 0.6093$.

From 1st three consecutive iterations,

$\Delta x_2 = x_3 - x_2 = 0.6093 - 0.5953$

$\Delta x_2 = 0.0140$

$\Delta^2 x_1 = \Delta(\Delta x_1) = \Delta(x_2 - x_1) = \Delta x_2 - \Delta x_1$

$= x_3 - x_2 - (x_2 - x_1)$

$= x_3 - 2x_2 + x_1$

$= 0.6093 - 2(0.5953) + 0.6667$

$= 0.6093 - 1.1906 + 0.6667$

$\Delta^2 x_1 = 0.0854$

By Aitken's method,

$$\alpha = x_{i+1} - \frac{(\Delta x_i)^2}{\Delta^2 x_{i-1}}$$

$$x_4 = x_{2+1} - \frac{(\Delta x_2)^2}{\Delta^2 x_{2-1}}$$

$$= x_3 - \frac{(\Delta x_2)^2}{\Delta^2 x_1}$$

$$= 0.6093 - \frac{(0.0140)^2}{0.0854}$$

$$= 0.6093 - \frac{0.000196}{0.0854}$$

$= 0.6093 - 0.0023$

$= 0.6070$ (correct to 3 decimal places)

$x_4 = 0.607$

$x_4 = 0.607$ corresponds to the 6^{th} iteration in the iteration method.

Exercise :

1) Find the root of the eqn $2x = \cos x + 3$ correct to three decimals using (i) iteration method (ii) Aitken's method.

soln:

i) Iteration method :

Let $f(x) = 2x - \cos x - 3$

$f(0) = 2(0) - \cos 0 - 3$

$= -1 - 3$

$f(0) = -4$

$f(\pi/2) = 2\pi/2 - \cos \pi/2 - 3$

$= \frac{6.2832}{2} - 0 - 3$

$= 3.1416 - 3$

$f(\pi/2) = 0.1416$

$\therefore$ one root of $f(x) = 0$ lies between 0 and $\pi/2$.

Now, $f(x) = 0$

$2x - \cos x - 3 = 0$

$2x = \cos x + 3$

$x = \frac{\cos x + 3}{2} = \phi(x)$

$$\phi'(x) = \frac{1}{2}(-\sin x) = \frac{-\sin x}{2}$$

$$|\phi'(x)| = \left|\frac{\sin x}{2}\right| < 1 \text{ for all } x \in (0, \pi/2)$$

If we take $x_0 = 0$, then the sequence of successive approximations $x_0, x_1, \ldots, x_n$ is convergent.

$$x_1 = \phi(x_0) = \phi(0) = \frac{1}{2}(\cos(0) + 3)$$

$$x_1 = \frac{1+3}{2} = 4/2 = 2$$

$$x_2 = \phi(x_1) = \phi(2) = \frac{1}{2}(\cos 2 + 3)$$

$$= \frac{-0.4161 + 3}{2} = \frac{2.5839}{2}$$

$$x_2 = 1.2920$$

$$x_3 = \phi(x_2) = \phi(1.2920) = \frac{1}{2}(\cos(1.2920) + 3)$$

$$= \frac{0.2752 + 3}{2} = \frac{3.2752}{3} =$$

$$x_3 = 1.6376$$

$$x_4 = \phi(x_3) = \cancel{\phi(1.6376)} = \frac{1}{2}(\cos(1.6376) + 3)$$

$$= \cancel{\frac{-0.0668 + 3}{2}} = \frac{3.4610}{2}$$

$$x_4 = \phi(x_3) = \phi(1.6376) = \frac{1}{2}(\cos(1.6376) + 3)$$

$$= \frac{-0.0668 + 3}{2} = \frac{2.9332}{2} = 1.4666$$

$$x_5 = \phi(x_4) = \phi(1.4666) = \frac{1}{2}(\cos(1.4666) + 3)$$

$$x_5 = \frac{0.1040 + 3}{2} = \frac{3.1040}{2} = 1.5520$$

$$x_6 = \phi(x_5) = \phi(1.5520) = \frac{1}{2}(\cos(1.5520)+3)$$

$$x_6 = \frac{0.0188+3}{2} = \frac{3.0188}{2} = 1.5094$$

$$x_7 = \phi(x_6) = \phi(1.5094) = \frac{1}{2}(\cos(1.5094)+3)$$

$$x_7 = \frac{0.0614+3}{2} = \frac{3.0614}{2} = 1.5307$$

$$x_8 = \phi(x_7) = \phi(1.5307) = \frac{1}{2}(\cos(1.5307)+3)$$

$$x_8 = \frac{0.0400+3}{2} = \frac{3.0400}{2} = 1.5200$$

$$x_9 = \phi(x_8) = \phi(1.5200) = \frac{1}{2}(\cos(1.5200)+3)$$

$$x_9 = \frac{0.0508+3}{2} = \frac{3.0508}{2} = 1.5254$$

$$x_{10} = \phi(x_9) = \phi(1.5254) = \frac{1}{2}(\cos(1.5254)+3)$$

$$x_{10} = \frac{0.0453+3}{2} = \frac{3.0453}{2} = 1.5226$$

$$x_{11} = \phi(x_{10}) = \phi(1.5226) = \frac{1}{2}(\cos(1.5226)+3)$$

$$x_{11} = \frac{0.0482+3}{2} = \frac{3.0482}{2} = 1.5241$$

$$x_{12} = \phi(x_{11}) = \phi(1.5241) = \frac{1}{2}(\cos(1.5241)+3)$$

$$x_{12} = \frac{0.0467+3}{2} = \frac{3.0467}{2} = 1.5233$$

$$x_{13} = \phi(x_{12}) = \phi(1.5233) = \frac{1}{2}(\cos(1.5233)+3)$$

$$x_{13} = \frac{0.0475+3}{2} = \frac{3.0475}{2} = 1.5238 \simeq 1.524$$

$$x_{14} = \phi(x_{13}) = \cancel{\frac{0.0470+3}{2}}\ \phi(1.5238) = \frac{1}{2}(\cos(1.5238)+3$$

$$x_{14} = \frac{0.0470 + 3}{2} = \frac{3.0470}{2} = 1.5235$$
$$\simeq 1.524$$

Here $x_{13} = x_{14}$, correct to three decimal places.

Hence, an approximate value of the required root is 1.524.

ii) Aitken's method:

From (1) we have first three iteration to be calculated

$$x_1 = 2$$
$$x_2 = 1.2920$$
$$x_3 = 1.6376$$

From 1st three consecutive iterations,

$$\Delta x_2 = x_3 - x_2 = 1.6376 - 1.2920$$
$$\Delta x_2 = 0.3456$$
$$\Delta^2 x_1 = \Delta(\Delta x_1) = x_3 - 2x_2 + x_1$$
$$= 1.6376 - 2(1.2920) + 2$$
$$= 1.6376 - 2.5840 + 2$$
$$\Delta^2 x_1 = 1.0536$$

By Aitken's method,

$$\alpha = x_{i+1} - \frac{(\Delta x_i)^2}{\Delta^2 x_{i-1}}$$

$$x_4 = x_{2+1} - \frac{(\Delta x_2)^2}{\Delta^2 x_{2-1}}$$

$$= x_3 - \frac{(\Delta x_2)^2}{\Delta^2 x_1}$$

$$= 1.6376 - \frac{(0.3456)^2}{1.0536}$$

$$= 1.6376 - \frac{0.1194}{1.0536}$$

$$= 1.6376 - 0.1133$$

$= 1.5243$ (correct to three decimal places)

$x_4 = 1.524$

Hence, the value of x_4 is 1.524

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iii. Bisection method (BOLZANA METHOD)

Pbm : 1

Find the real root of the equation $x^3 - 3x + 1 = 0$ lying between 1 and 2 correct to three places of decimals by using bisection method.

Soln:

$$f(x) = x^3 - 3x + 1$$

$$f(1) = 1 - 3 + 1 = -1 < 0$$

$$f(2) = 8 - 6 + 1 = 3 > 0$$

$\therefore$ one root of $f(x) = 0$ lies between 1 and 2.

Let $a = 1,\ b = 2$.

1^{st} approximation, $x_1 = \frac{a+b}{2} = \frac{1+2}{2}$

$\left(x_0 = \frac{a+b}{2} \text{ (midpt)} \right)$ $= 3/2$

$x_1 = 1.5$

Now, $f(1.5) = (1.5)^3 - 3(1.5) + 1$

$= 3.375 - 4.5 + 1$

$f(1.5) = -0.125$ (-ve)

Also, $f(2) = 3$ (+ve)

Hence, the root lies between 1.5 and 2

Let $a = 1.5$, $b = 2$

2^{nd} approximation, $x_2 = \frac{1.5+2}{2} = \frac{3.5}{2}$

$x_2 = 1.75$

Now, $f(1.75) = (1.75)^3 - 3(1.75) + 1$

$= 5.3594 - 5.25 + 1$

$f(1.75) = 1.1094$ (+ve)

Also, $f(1.5) = -0.125$ (-ve)

~~3rd app~~ Hence, the root lies between 1.5 and 1.75.

Let $a = 1.5$, $b = 1.75$

3^{rd} approximation, $x_3 = \frac{1.5+1.75}{2} = \frac{3.25}{2}$

$x_3 = 1.625$

Now, $f(1.625) = (1.625)^3 - 3(1.625) + 1$

$= 4.2910 - 4.875 + 1$

$f(1.625) = 0.4160$ (+ve)

Now, $f(1.5) =$ (-ve)

Hence, the root lies between 1.5 and 1.625.

It can be calculated as.

i	a	b	$x_i = \frac{a+b}{2}$	$f(x_i)$
1	1 (-ve)	2 (+ve)	$x_1 = 1.5$	-0.1250 (-ve)
2	1.5 (-ve)	2	$x_2 = 1.75$	1.1094 (+ve)
3	1.5	1.75 (+ve)	$x_3 = 1.625$	0.4160 (+ve)
4	1.5	1.625	$x_4 = \frac{1.5+1.625}{2}$ $x_4 = 1.5625$	$f(1.5625) = (1.5625)^3 - 3(1.5625) + 1$ $= 3.8147 - 4.6875 + 1$ $= 0.1272$ (+ve)
5	1.5	1.5625	$x_5 = \frac{1.5+1.5625}{2}$ $x_5 = 1.5313$	$f(1.5313) = 3.5907 - 4.5939 + 1$ $= -0.0032$ (-ve)
6	1.5313	1.5625	$x_6 = \frac{3.0938}{2}$ $x_6 = 1.5469$	$f(1.5469) = 3.7016 - 4.6407 + 1$ $= 0.0609$ (+ve)
7	1.5313	1.5469	$x_7 = \frac{3.0782}{2}$ $x_7 = 1.5391$	$f(1.5391) = 3.6459 - 4.6173 + 1$ $= 0.0286$ (+ve)
8.	1.5313	1.5391	$x_8 = \frac{3.0704}{2}$ $x_8 = 1.5352$	$f(1.5352) = 3.6182 - 4.6056 + 1$ $= 0.0126$ (+ve)
9	1.5313	1.5352	$x_9 = \frac{3.0665}{2}$ $x_9 = 1.5333$	$f(1.5333) = 3.6048 - 4.5999 + 1$ $= 0.0049$ (+ve)
10	1.5313	1.5333	$x_{10} = \frac{3.0646}{2}$ $x_{10} = 1.5323$	$f(1.5323) = 3.5978 - 4.5969 + 1$ $= 0.0009$ (+ve)
11	1.5313	1.5323	$x_{11} = \frac{3.0636}{2}$ $x_{11} = 1.5318$	

Here, $x_{10} = x_{11} = 1.532$ correct to three decimal places.

Hence, 1.532 is the approximate value of the required root.

Pbm: 2.

Find a real root of $x^3 - x - 11 = 0$ by using bisection method.

Soln:

Let $f(x) = x^3 - x - 11$

$f(0) = 0 - 0 - 11 = -11$

$f(1) = 1 - 1 - 11 = -11$

$f(2) = 8 - 2 - 11 = -5 < 0 \quad (-ve)$

$f(3) = 27 - 3 - 11 = 13 > 0 \quad (+ve)$

$\therefore$ one root of $f(x) = 0$ lies between 2 and 3.

Let $\Rightarrow a = 2$ & $b = 3$

1st approximation, $x_1 = \frac{a+b}{2} = \frac{2+3}{2} = 5/2 = 2.5$

(Midpoint $x = \frac{a+b}{2}$)

$f(2.5) = (2.5)^3 - 2.5 - 11$

$= 15.625 - 2.5 - 11$

$f(2.5) = 2.125 \quad (+ve) \quad ([f(2) \to (-ve)])$

Hence, the root lies between 2 and 2.5.

Let $a = 2$ and $b = 2.5$.

2nd approximation, $x_2 = \frac{2+2.5}{2} = \frac{4.5}{2} = 2.25$

$f(2.25) = (2.25)^3 - 2.25 - 11$

$= 11.3906 - 2.25 - 11$

$f(2.25) = -1.8594$ (-ve) [Also, $f(2.5) \to$ +ve]

Hence, The root lies between 2.25 and 2.5

i	a	b	$x_i = \frac{a+b}{2}$	$f(x_i)$
1.	2	3	$x_1 = 5/2 = 2.5$	$f(2.5) = 15.625 - 2.5 - 11 = 2.125$
2.	2	2.5	$x_2 = \frac{4.5}{2} = 2.25$	$f(2.25) = 11.3906 - 2.25 = -1.8594$
3.	2.25	2.5	$x_3 = \frac{4.75}{2} = 2.375$	$f(2.375) = 13.3965 - 2.3 = 0.0215$
4.	2.25	2.375	$x_4 = \frac{4.625}{2} = 2.3125$	$f(2.3125) = 12.3665 - 2.3 - 11 = -0.9460$
5.	2.3125	2.375	$x_5 = \frac{4.6875}{2} = 2.3438$	$f(2.3438) = 12.8754 - 2. - 11 = -0.4684$
6.	2.3438	2.375	$x_6 = \frac{4.7188}{2} = 2.3594$	$f(2.3594) = 13.1342 - 2. - 11 = -0.2252$
7.	2.3594	2.375	$x_7 = \frac{4.7344}{2} = 2.3672$	$f(2.3672) = 13.2649 - 2.3 - 11 = -0.1023$
8.	2.3672	2.375	$x_8 = \frac{4.7422}{2} = 2.3711$	$f(2.3711) = 13.3306 - 2.371 = -0.0405$
9.	2.3711	2.375	$x_9 = \frac{4.7461}{2} = 2.3731$	$f(2.3731) = 13.3644 - 2.373 = -0.0087$
10.	2.3731	2.375	$x_{10} = \frac{4.7481}{2} = 2.3741$	$f(2.3741) = 13.3813 - 2.374 = 0.0072$

11.	2.3731	2.3741	$x_{11} = \frac{4.7472}{2} = 2.3736$	$f(2.3736) = 13.3728 - 2.3736 - 11$ $= -0.0008$
12.	2.3736	2.3741	$x_{12} = \frac{4.7477}{2} = 2.3739$	$f(2.3739) = 13.3779 - 2.3739 - 11$ $= 0.004$
13.	2.3736	2.3739	$x_{13} = \frac{4.7475}{2} = 2.3738$	$f(2.3738) = 13.3762 - 2.3738 - 11$ $= 0.0024$
14.	2.3736	2.3738	$x_{14} = \frac{4.7474}{2} = 2.3737$	$f(2.3737) = 13.3745 - 2.3737 - 11$ $= 0.0008$
15.	2.3736	2.3737	$x_{15} = \frac{4.7473}{2} = 2.3737$	$f(2.3737) = 13.3745 - 2.3737 - 11$ $= 0.0008$

From the above table, we have

$x_{14} = x_{15} = 2.3737$ (four decimal places)

$x_{14} = x_{15} = 2.374$ (correct to 3 decimal places)

Hence, the value of the root upto three places of decimals is 2.374 and upto four places of decimals is 2.3737.

Pbm: 3

Find the positive root of $x \log_{10} x = 1.2$ using the bisection method in four iteration

soln:

Let $f(x) = x \log_{10} x - 1.2$.

$f(0) = 0 - 1.2 = -1.2$

$f(1) = 0 - 1.2 = -1.2$

$f(2) = 2 \log_{10}(2) - 1.2$

$= 2(0.30103) - 1.2$

$f(2) = 0.6021 - 1.2 = -0.5979 < 0$

$f(3) = 3 \log_{10}(3) - 1.2$

$= 3(0.47712) - 1.2$

$= 1.43136 - 1.2$

$= 0.23136$

$f(3) = 0.2314 > 0$

$\therefore$ one root of $f(x) = 0$ lies between 2 and 3.

Let $a = 2$, $b = 3$

1st approximation, $x_1 = \frac{a+b}{2} = \frac{2+3}{2}$

$x_1 = \frac{5}{2} = 2.5$

$f(2.5) = 2.5 \log_{10}(2.5) - 1.2$

$= 2.5(0.3979) - 1.2$

$= 0.99475 - 1.2$

$= -0.20525$

$f(2.5) = -0.2053 < 0$

$f(3) > 0$

Hence, the root lies between 2.5 and 3.

Let $a = 2.5$, $b = 3$

2^{nd} approximation, $x_2 = \frac{2.5+3}{2} = \frac{5.5}{2} = 2.75$

$f(2.75) = 2.75 \log_{10}(2.75) - 1.2$

$= 2.75 \times 0.4393 - 1.2$

$= 1.2080 - 1.20$

$f(2.75) = 0.008 > 0$

$f(2.5) < 0$

Hence, the root lies between 2.5 and 2.75.

Let $a = 2.5$ and $b = 2.75$

3^{rd} approximation, $x_3 = \frac{2.5+2.75}{2} = \frac{5.25}{2}$

$x_3 = 2.625$

$f(2.625) = 2.625 \log_{10}(2.625) - 1.2$

$= 2.625 \times 0.4191 - 1.2$

$= 1.1001 - 1.2$

$= -0.0999$

$= -0.100$

$f(2.625) = -0.10 < 0$

$f(2.75) > 0$

Hence, the root lies between 2.625 and 2.75.

Let $a = 2.625$ & $b = 2.75$

4^{th} approximation, $x_4 = \frac{2.625+2.75}{2} = \frac{5.375}{2}$

$x_4 = 2.6875$

Hence, the approximate value of the required root of the fourth iteration is 2.6875.

Exercises.

1) Find a root of the eqn $x^3 - 4x - 9 = 0$ using the bisection method in four iterations.

soln:

Let $f(x) = x^3 - 4x - 9$

$f(0) = 0 - 0 - 9 = -9$

$f(1) = 1 - 4 - 9 = -12$

$f(2) = 8 - 8 - 9 = -9 < 0$

$f(3) = 27 - 12 - 9 = 6 > 0$

$\therefore$ One root of $f(x) = 0$ lies between 2 and 3.

Let $a = 2,\ b = 3$.

1st approximation, $x_1 = \frac{a+b}{2} = \frac{2+3}{2} = \frac{5}{2} = 2.5$

$f(2.5) = (2.5)^3 - 4(2.5) - 9$

$= 15.265 - 10 - 9$

$f(2.5) = -3.735 < 0$

$f(3) > 0$

Hence, the root lies between 2.5 and 3.

Let $a = 2.5,\ b = 3$

2nd approximation, $x_2 = \frac{2.5+3}{5} = \frac{5.5}{5} = 2.75$

$f(2.75) = (2.75)^3 - 4(2.75) - 9$

$= 20.7969 - 11 - 9$

$f(2.75) = 0.7969 > 0$

$f(2.5) < 0$

Hence, the root lies between 2.5 and 2.75.

Let $a = 2.5$, $b = 2.75$

3^{rd} approximation, $x_3 = \frac{2.5+2.75}{2} = \frac{5.25}{2}$

$x_3 = 2.625$

$f(2.625) = (2.625)^3 - 4(2.625) - 9$

$= 18.0879 - 10.5 - 9$

$f(2.625) = -1.4121 < 0$

$f(2.75) > 0$

Hence, the root lies between 2.625 & 2.75

Let $a = 2.625$ & $b = 2.75$

4^{th} approximation, $x_4 = \frac{2.625+2.75}{2}$

$= \frac{5.375}{2}$

$x_4 = 2.6875$

Hence, the approximate value of the required root of the fourth iteration is 2.6875.

3) Find a root of the following eqns using bisection method.

i) $x^3 - x - 1 = 0$

Let $f(x) = x^3 - x - 1$

$f(0) = -1 < 0$

$f(1) = -1 < 0$

$f(2) = 5 > 0$

$\therefore$ one root of $f(x) = 0$ lies between 1 and 2.

It can be calculated as

i	a	b	$x_i = \frac{a+b}{2}$	$f(x_i)$
1	1	2	$x_1 = \frac{3}{2} = 1.5$	$f(x_i) = f(1.5) = (1.5)^3 - (1.5) - 1$ $= 3.375 - 1.5 - 1$ $= 0.875 > 0$
2	1	1.5	$x_2 = \frac{2.5}{2} = 1.25$	$f(1.25) = 1.9531 - 1.25$ $= -0.2969 < 0$
3	1.25	1.5	$x_3 = \frac{2.75}{2} = 1.375$	$f(1.375) = 2.5996 - 1.375$ $= 0.2246 > 0$
4.	1.25	1.375	$x_4 = \frac{2.625}{2} = 1.3125$	$f(1.3125) = 2.2610 - 1.3125 - 1$ $= -0.0515 < 0$
5.	1.3125	1.375	$x_5 = \frac{2.6875}{2} = 1.3438$	$f(1.3438) = 2.4266 - 1.3438$ $= 0.0828 > 0$
6.	1.3125	1.3438	$x_6 = \frac{2.6563}{2} = 1.3282$	$f(1.3282) = 2.3431 - 1.3282 - 1$ $= 0.0149 > 0$
7.	1.3125	1.3282	$x_7 = \frac{2.6407}{2} = 1.3204$	$f(1.3204) = 2.3021 - 1.3204 - 1$ $= -0.0183 < 0$
8	1.3204	1.3282	$x_8 = \frac{2.6486}{2} = 1.3243$	$f(1.3243) = 2.3225 - 1.3243 - 1$ $= -0.0018 < 0$
9	1.3243	1.3282	$x_9 = \frac{2.6525}{2} = 1.3263$	$f(1.3263) = 2.3331 - 1.3263 - 1$ $= 0.0068 > 0$
10	1.3243	1.3263	$x_{10} = \frac{2.6506}{2} = 1.3253$	$f(1.3253) = 2.3278 - 1.3253 - 1$ $= 0.0025 > 0$

11	1.3243	1.32
12.	1.3243	1.32
13.	1.3246	1.2
14.	1.3247	1.3
15.	1.3247	1.

places of

ii) $x^3 - 9x + 1 = 0$

2 and

L

11	1.3243	1.3253	$x_{11} = \frac{2.6496}{2} = 1.3248$	$f(1.3248) = 2.3251 - 1.3248 - 1 = 0.0003 > 0$
12	1.3243	1.3248	$x_{12} = \frac{2.6491}{2} = 1.3246$	$f(1.3246) = 2.3243 - 1.3246 - 1 = -0.0005 < 0$
13	1.3246	1.3248	$x_{13} = \frac{2.6494}{2} = 1.3247$	$f(1.3247) = 2.3246 - 1.3247 - 1 = -0.0001 < 0$
14	1.3247	1.3248	$x_{14} = \frac{2.6495}{2} = 1.3248$	$f(1.3248) = 2.3251 - 1.3248 - 1 = 0.0003 > 0$
15	1.3247	1.3248	$x_{15} = \frac{2.6495}{2} = 1.3248$	—

Hence $x_{14} = x_{15} = 1.3248$ correct to four places of decimals.

Hence the required root is 1.3248

5) $x^3 - 9x + 1 = 0$

Let $f(x) = x^3 - 9x + 1$

$f(0) = 1 > 0$

$f(1) = -7 < 0$

$f(2) = -9 < 0$

$f(3) = 13 > 0$

$\therefore$ one root of $f(x) = 0$ lies between 2 and 3.

Let $a = 2,\ b = 3$

$$x_1 = \frac{2+3}{2} = \frac{5}{2} = 2.5$$

$$f(2.5) = (2.5)^3 - 9(2.5) + 1$$

$$= 15.265 - 22.5 + 1$$

$$f(2.5) = -6.235 < 0$$

It can be calculated as

i	a	b	$x_i = \frac{a+b}{2}$	$f(x_i)$
1.	2	3	$x_1 = 5/2 = 2.5$	$f(2.5) = 15.625 - 22.50 + 1 = -5.875 < 0$
2.	2.5	3	$x_2 = \frac{5.5}{2} = 2.75$	$f(2.75) = 20.7969 - 24.75 + 1 = -2.9531 < 0$
3.	2.75	3	$x_3 = \frac{5.75}{2} = 2.875$	$f(2.875) = 23.7637 - 25.8 + 1 = -1.1113 < 0$
4.	2.875	3	$x_4 = \frac{5.875}{2} = 2.9375$	$f(2.9375) = 25.3474 - 26. + 1 = -0.0901 < 0$
5.	2.9375	3	$x_5 = \frac{5.9375}{2} = 2.9688$	$f(2.9688) = 26.1663 - 26.7 + 1 = 0.4471 > 0$
6.	2.9375	2.9688	$x_6 = \frac{5.9063}{2} = 2.9532$	$f(2.9532) = 25.7560 - 26.5 + 1 = 0.1772 > 0$
7.	2.9375	2.9532	$x_7 = \frac{5.8907}{2} = 2.9454$	$f(2.9454) = 25.5525 - 26.5 + 1 = 0.0439 > 0$
8.	2.9375	2.9454	$x_8 = \frac{5.8829}{2} = 2.9415$	$f(2.9415) = 25.4511 - 26.47 = -0.0224 < 0$
9.	2.9415	2.9454	$x_9 = \frac{5.8869}{2} = 2.9435$	$f(2.9435) = 25.5030 - 26.4 + 1 = 0.0115 > 0$
10.	2.9415	2.9435	$x_{10} = \frac{5.885}{2} = 2.9425$	$f(2.9425) = 25.4771 - 2 = -0.0054 < 0$
11.	2.9425	2.9435	$x_{11} = \frac{5.886}{2} = 2.943$	$f(2.943) = 25.4901 - 26.48 = 0.0031 > 0$

12	2.9425	2.943	$x_{12} = \frac{5.8855}{2} = 2.9428$	$f(2.9428) = 25.4849 - 26.4852 + 1 = -0.0003 < 0$
13	2.9428	2.943	$x_{13} = \frac{5.8858}{2} = 2.9429$	$f(2.9429) = 25.4875 - 26.4861 + 1 = 0.0014 > 0$
14	2.9428	2.9429	$x_{14} = \frac{5.8857}{2} = 2.94285$, $x_{14} = 2.9428$	$f(2.9428) = 25.4849 - 26.4852 + 1 = -0.0003 < 0$
15)	2.9428	2.9429	$x_{15} = \frac{5.8857}{2} = 2.94285$, $x_{15} = 2.9428$	-

Hence $x_{14} = x_{15} = 2.9428$ correct to four decimal places.

Hence the required root is 2.9428.

5) Find the positive root of $x - \cos x = 0$ by bisection method.

Let $f(x) = x - \cos x$

$f(0) = 0 - \cos 0 = 0 - 1 = -1 < 0$

$f(1) = 1 - \cos 1 = 1 - 0.5403 = 0.4597 > 0$

$\therefore$ one root of $f(x) = 0$ lies between 0 and 1.

It can be calculated as.

i	a	b	$x_i = \frac{a+b}{2}$	$f(x_i)$
1.	0	1	$x_1 = \frac{1}{2} = 0.5$	$f(0.5) = 0.5 - 0.8776 = -0.3776 < 0$
2.	0.5	1	$x_2 = \frac{1.5}{2} = 0.75$	$f(0.75) = 0.75 - 0.7317 = 0.0183 > 0$
3.	0.5	0.75	$x_3 = \frac{1.25}{2} = 0.625$	$f(0.625) = 0.625 - 0.8110 = -0.1860 < 0$

4.	0.625	0.75	$x_4 = \frac{1.375}{2} = 0.6875$	$f(0.6875) = 0.6875 - 0.7728 = -0.0853 < 0$
5.	0.6875	0.75	$x_5 = \frac{1.4375}{2} = 0.7188$	$f(0.7188) = 0.7188 - 0.7526 = -0.0338 < 0$
6.	0.7188	0.75	$x_6 = \frac{1.4688}{2} = 0.7344$	$f(0.7344) = 0.7344 - 0.7422 = -0.0078 < 0$
7.	0.7344	0.75	$x_7 = \frac{1.4844}{2} = 0.7422$	$f(0.7422) = 0.7422 - 0.7370 = 0.0052 > 0$
8.	0.7344	0.7422	$x_8 = \frac{1.4766}{2} = 0.7383$	$f(0.7383) = 0.7383 - 0.7396 = -0.0013 < 0$
9.	0.7383	0.7422	$x_9 = \frac{1.4805}{2} = 0.7403$	$f(0.7403) = 0.7403 - 0.7383 = 0.0002 > 0$
10.	0.7383	0.7403	$x_{10} = \frac{1.4786}{2} = 0.7393$	$f(0.7393) = 0.7393 - 0.7389 = 0.0004 > 0$
11.	0.7383	0.7393	$x_{11} = \frac{1.4776}{2} = 0.7388$	$f(0.7388) = 0.7388 - 0.7393 = -0.0005 < 0$
12	0.7388	0.7393	$x_{12} = \frac{1.4781}{2} = 0.7391$	$f(0.7391) = 0.7391 - 0.7390 = 0.0001 > 0$
13	0.7388	0.7391	$x_{13} = \frac{1.4779}{2} = 0.7390$	—

Here, $x_{12} = x_{13}$ correct to three decimal places.

Hence, the required root is 0.739.

Exercises

1) $x^3 - 2x - 50 =$

Soln:

Let

4.

1st

f(

and

Regula falsi Method

(Method of false position)

Exercises.

1) $x^3 - 2x - 50 = 0$

Soln:

Let $f(x) = x^3 - 2x - 50$

$f(0) = -50 < 0$

$f(1) = -51 < 0$

$f(2) = 8 - 8 - 50 = -50 < 0$

$f(3) = 27 - 6 - 50 = -29 < 0$

$f(4) = 64 - 8 - 50 = 6 > 0$

$\therefore$ one root of $f(x) = 0$ lies between 3 and 4.

$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

1st approximation,

Let $a = 3,\ b = 4,\ f(a) = -29,\ f(b) = 6$

$$x_1 = \frac{3(6) - 4(-29)}{6 + 29} = \frac{18 + 116}{35}$$

$$x_1 = \frac{134}{35} = 3.8286$$

$f(x_1) = f(3.8286) = 56.1203 - 7.6572 - 50$

$f(3.8286) = -1.5369 < 0$

$\therefore$ The root lies between $f(3.8286) < 0$ and $f(4) > 0$.

2nd approximation,

Let $a = 3.8286$, $b = 4$, $f(a) = -1.5369$, $f(b) = 6$

$$x_2 = \frac{3.8286(6) - 4(-1.5369)}{6-(-1.5369)}$$

$$x_2 = \frac{22.9716 + 6.1476}{7.5369} = 3.8636$$

$f(x_2) = f(3.8636) = 57.6735 - 7.7272 - 50$

$f(3.8636) = -0.0537 < 0$

$\therefore$ The root lies between $f(3.8636) < 0$ and $f(4) > 0$.

Let $a = 3.8636$, $b = 4$, $f(a) = -0.0537$, $f(b) = 6$

Third approximation is

$$x_3 = \frac{3.8636(6) - 4(-0.0537)}{6-(-0.0537)}$$

$$x_3 = \frac{23.1816 + 0.2148}{6.0537} = 3.8648$$

$f(x_3) = f(3.8648) = 57.7273 - 7.7296 - 50$

$f(3.8648) = -0.0023 < 0$

$\therefore$ The root lies between $f(3.8648) < 0$ and $f(4) > 0$

4th approximation, let $a = 3.8648$, $b = 4$,

$f(a) = -0.0023$, $f(b) = 6$

$$x_4 = \frac{3.8648(6) - 4(-0.0023)}{6-(-0.0023)}$$

$$= \frac{23.1888 + 0.0092}{6.0023}$$

$$x_4 = \frac{23.198}{6.0023} = 3.8649$$

$x_3 = x_4 = 3.865$ correct to three decimal places.

Hence, the required root is 3.865.

3) $xe^x - 2 = 0$

Soln:

Let $f(x) = xe^x - 2$

$f(0) = 0 - 2 = -2 < 0$

$f(1) = e^1 - 2 = 2.7183 - 2 = 0.7183 > 0$

$\therefore$ one root of $f(x) = 0$ lies between 0 and 1.

Let $a = 0$, $b = 1$, $f(a) = -2$, $f(b) = 0.7183$

1st approximation,

$$x_1 = \frac{0(0.7183) - 1(-2)}{0.7183 + 2} = \frac{2}{2.7183} = 0.7358$$

$f(x_1) = f(0.7358) = 1.5358 - 2 = -0.4642 < 0$

$\therefore$ The root lies between $f(0.7358) < 0$ and $f(1) > 0$

Let $a = 0.7358$, $b = 1$, $f(a) = -0.4642$, $f(b) = 0.7183$

2nd approximation,

$$x_2 = \frac{0.7358(0.7183) - 1(-0.4642)}{0.7183 + 0.4642}$$

$$x_2 = \frac{0.5285 + 0.4642}{1.1825} = 0.8395$$

$f(x_2) = f(0.8395) = 1.9436 - 2 = -0.0564 < 0$

$\therefore$ The root lies between $f(0.8395) < 0$ and $f(1) > 0$.

Let $a = 0.8395$, $b = 1$, $f(a) = -0.0564$,

$f(b) = 0.7183$

3rd approximation,

$$x_3 = \frac{0.8395(0.7183) - 1(-0.0564)}{0.7183 + 0.0564}$$

$$x_3 = \frac{0.6030 + 0.0564}{0.7747} = 0.8512$$

$f(x_3) = f(0.8512) = 1.9939 - 2 = -0.0061 < 0$

The root lies between $f(0.8512) < 0$ and $f(1) > 0$.

Let $a = 0.8512$, $b = 1$, $f(a) = -0.0061$, $f(b) = 0.7183$

4th approximation,

$$x_4 = \frac{0.8512(0.7183) - 1(-0.0061)}{0.7183 + 0.0061}$$

$$x_4 = \frac{0.6114 + 0.0061}{0.7244} = 0.8524$$

$f(x_4) = f(0.8524) = 1.9991 - 2 = -0.0009 < 0$

The root lies between $f(0.8524) < 0$ and $f(1) > 0$

Let $a = 0.8524$, $b = 1$, $f(a) = -0.0009$, $f(b) = 0.7183$.

5th approximation,

$$x_5 = \frac{0.8524(0.7183) - 1(-0.0009)}{0.7183 + 0.0009}$$

$$x_5 = \frac{0.6123 + 0.0009}{0.7192} = 0.8526$$

$f(x_5) = f(0.8526) = 1.9999 - 2 = -0.0001 < 0$

$\therefore$ The root lies between $f(0.8526) < 0$ and $f(1) > 0$

Let $a = 0.8526$, $b = 1$, $f(a) = -0.0001$,
$f(b) = 0.7183$

6^{th} approximation,

$$x_6 = \frac{0.8526(0.7183) - 1(-0.0001)}{0.7183 + 0.0001}$$

$$x_6 = \frac{0.6124 + 0.0001}{0.7184} = 0.8526$$ 2

Here $x_5 = x_6$ correct to four decimal places.

Hence, the required root is 0.8526

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3. 2) $x^3 - 4x + 1 = 0$ 3.

Let $f(x) = x^3 - 4x + 1$

$f(0) = 0 - 0 + 1 = 1 > 0$

$f(1) = 1 - 4 + 1 = -2 < 0$

$\therefore$ one root $f(x)=0$ lies between 0 and 1.

Let $a = 0$, $b = 1$, $f(a) = 1$, $f(b) = -2$.

1^{st} approximation, $x_1 = \dfrac{0(-2) - 1(1)}{-2 - 1} = \dfrac{-1}{-3}$

$x_1 = \dfrac{1}{3} = 0.3333$.

$f(x_1) = f(0.3333) = 0.0370 - 1.3332 + 1$

$f(0.3333) = -0.2962 < 0$

$\therefore$ The root lies between $f(0.3333) < 0$ and 3

$f(0) > 0$

Let $a = 0.3333$, $b = 0$

$f(a) = -0.2962$, $f(b) = +1$

2nd approximation,

$$x_2 = \frac{0.3333(1) - 0(-0.2962)}{1 + 0.2962}$$

$$x_2 = \frac{0.3333 + 0}{1.2962} = 0.2571$$

$f(x_2) = f(0.2571) = 0.0170 - 1.0284 + 1$

$f(0.2571) = -0.0114 < 0$

$\therefore$ The root lies between $f(0.2571) < 0$ and $f(0) > 0$.

Let $a = 0.2571$, $b = 0$, $f(a) = -0.0114$, $f(b) = 1$

Third approximation,

$$x_3 = \frac{0.2571(1) - 0(-0.0114)}{1 + 0.0114}$$

$$x_3 = \frac{0.2571 + 0}{1.0114} = 0.2542$$

$f(x_3) = f(0.2542) = 0.0164 - 1.0168 + 1$

$f(0.2542) = -0.0004 < 0$

$\therefore$ The root lies between $f(0.2542) < 0$ and $f(0) > 0$.

Let $a = 0.2542$, $b = 0$, $f(a) = -0.0004$, $f(b) = 1$

Fourth approximation, $x_4 = \dfrac{0.2542(1) - 0(-0.0004)}{1 + 0.0004}$

$$x_4 = \frac{0.2542}{1.0004} = 0.2541$$

$x_3 = x_4 = 0.254$ (correct to three decimals)

Hence the required root is 0.254.

4) $x \log_{10} x = 1.2$

Soln:

Let $f(x) = x \log_{10} x - 1.2$

$f(0) = 0 \log_{10}(0) - 1.2 = -1.2$

$f(1) = \log_{10}(1) - 1.2 = -1.2$

$f(2) = 2 \log_{10}(2) - 1.2 = 2(0.3010) - 1.2$

$f(2) = 0.602 - 1.2 = -0.598$

$f(3) = 3 \log_{10}(3) - 1.2 = 3(0.4771) - 1.2$

$f(3) = 1.4313 - 1.2 = 0.2313$

$\therefore$ one root of $f(x) = 0$ lies between 2 and 3.

Let $a = 2$, $b = 3$, $f(a) = -0.598$, $f(b) = 0.2313$.

1st approximation,

$$x_1 = \frac{2(0.2313) - 3(-0.598)}{0.2313 + 0.598} = \frac{0.4626 + 1.794}{0.8293}$$

$x_1 = 2.7211$.

$f(x_1) = f(2.7211) = 2.7211 \log_{10}(2.7211) - 1.2$

$= 2.7211(0.4347) - 1.2$

$f(2.7211) = 1.1829 - 1.2 = -0.0171 < 0$

$\therefore$ the root lies between $f(2.7211) < 0$ and $f(3) > 0$.

Let $a = 2.7211$, $b = 3$, $f(a) = -0.0171$, $f(b) = 0.2313$

2nd approximation, $x_2 = \dfrac{2.7211(0.2313) - 3(-0.0171)}{0.2313 + 0.0171}$

$$x_2 = \frac{0.6294 + 0.0513}{0.2484} = 2.7403$$

$f(x_2) = f(2.7403) = 2.7403 \log_{10}(2.7403) - 1.2$

$= 2.7403(0.4378) - 1.2$

$f(2.7403) = 1.1997 - 1.2 = -0.0003 < 0$

$\therefore$ The root lies between $f(2.7403) < 0$ and $f(3) > 0$.

Let $a = 2.7403$, $f(a) = -0.0003$

$b = 3$, $f(b) = 0.2313$

3rd approximation, $x_3 = \dfrac{2.7403(0.2313) - 3(-0.0003)}{0.2313 + 0.0003}$

$= \dfrac{0.6338 + 0.0009}{0.2316}$

$x_3 = 2.7405$

$f(2.7405) = 2.7405 \log_{10}(2.7405) - 1.2$

$= 2.7405(0.4378) - 1.2 = 1.1998 - 1.2$

$f(2.7405) = -0.0002 < 0$

$\therefore$ The root lies between $f(2.7405) < 0$ and $f(3) > 0$

Let $a = 2.7405$, $b = 3$, $f(a) = -0.0002$,

$f(b) = 0.2313$.

4th approximation,

$x_4 = \dfrac{2.7405(0.2313) - 3(-0.0002)}{0.2313 + 0.0002}$

$x_4 = \dfrac{0.6339 + 0.0006}{0.2315} = 2.7408$

Here $x_3 = x_4$ correct to three decimals

Hence, the required root is 2.741

Newton's - Raphson Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad (n = 0, 1, 2, \cdots)$$

Exercise:

Find a positive root of each of the following equations using Newton's-Raphson method.

1) $x^3 + x - 1 = 0$.

Soln:

Let $f(x) = x^3 + x - 1$

$f'(x) = 3x^2 + 1$

$f(0) = 0 + 0 - 1 = -1 < 0$

$f(1) = 1 + 1 - 1 = 1 > 0$

$\therefore$ The root lies between 0 and 1

Take $x_0 = 0$

When, $n = 0$, 1st approximation,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \qquad [f'(0) = 1]$$

$$x_1 = 0 - \frac{-1}{1} = 0 + 1 = 1$$

Take, $x_1 = 1$, $f(x_1) = f(1) = 1$, $f'(x_1) = 3 + 1 = 4$

2nd approximation, $x_2 = 1 - \frac{1}{4} = \frac{3}{4}$

$x_2 = 0.75$

$$x_2 = \cancel{x_1} - \frac{f(x_1)}{f'(x_1)}$$

$$= \cancel{1} - \frac{1}{4} =$$

Take $x_2 = 0.75$, $f(x_2) = (0.75)^3 + 0.75 - 1$

$$= 0.4219 + 0.75 - 1$$

$$f(x_2) = 0.1719$$

$$f'(x_2) = 3(0.75)^2 + 1 = 3(0.5625) + 1$$

$$= 1.6875 + 1$$

$$f'(x_2) = 2.6875$$

Third approximation, $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

$$= 0.75 - \frac{0.1719}{2.6875} = 0.75 - 0.0640$$

$$x_3 = 0.6860$$

Take $x_3 = 0.6860$, $f(x_3) = (0.6860)^3 + 0.6860 - 1$

$$= 0.3228 + 0.6860 - 1$$

$$f(x_3) = 0.0088$$

$$f'(x_3) = 3(0.6860)^2 + 3 = 3(0.4706) + 3$$

$$= 1.4118 + 1$$

$$f'(x_3) = 2.4118$$

Fourth approximation, $x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$

$$= 0.6860 - \frac{0.0088}{2.4118}$$

$$\cancel{= 0.6860 - 2.5799}$$

$$= 0.6860 - 0.0036$$

$$x_4 = 0.6824$$

Take $x_4 = 0.6824$, $f(x_4) = (0.6824)^3 + 0.6824 - 1$

$$= 0.3178 + 0.6824 - 1$$

$$f(x_4) = 0.0002$$

$$f'(x_4) = 3(0.6824)^2 + 1 = 3(0.4657) + 1$$

$$f'(x_4) = 0.4657 + 1 = 1.4657$$

fifth approximation, $x_5 = x_4 - \frac{f(x_4)}{f'(x_4)}$

$$= 0.6824 - \frac{0.0002}{1.4657}$$

$$= 0.6824 - 0.0001$$

$$x_5 = 0.6823$$

Take $x_5 = 0.6823$, $f(x_5) = (0.6823)^3 + 0.6823 - 1$

$$= 0.3176 + 0.6823 - 1$$

$$f(x_5) = 0.0001$$

$$f'(x_5) = 3(0.6823)^2 + 1 = 3(0.4655) + 1$$

$$= 1.3965 + 1$$

$$f(x_5) = 2.3965$$

Sixth approximation, $x_6 = x_5 - \frac{f(x_5)}{f'(x_5)}$

$$= 0.6823 - \frac{0.0001}{2.3965}$$

$$= 0.6823 - 0.0000$$

$$x_6 = 0.6823$$

Here, $x_5 = x_6$.

Hence, the required root is 0.6823.

Newton's Raphson Method.

2) $x^3 - 5x - 6 = 0$.

$f(x) = x^3 - 5x - 6$.

$f'(x) = 3x^2 - 5$.

$f(0) = 0 - 0 - 6 = -6 < 0$

$f(1) = 1 - 5 - 6 = -10 < 0$

$f(2) = 8 - 10 - 6 = -8 < 0$

$f(3) = 27 - 15 - 6 = 5 > 0$

one root of $f(x) = 0$ lies between 2 and 3.

Take $x_0 = 2$, $f'(2) = 3(4) - 5 = 12 - 5 = 7$.

when $n = 0$, $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$x_1 = 2 - \frac{(-8)}{7} = 2 + 8/7 = 2 + 1.1429$

$x_1 = 3.1429$

$f(x_1) = f(3.1429) = 31.045 - 15.7145 - 6 = 9.3305$

$f'(x_1) = f'(3.1429) = 29.6334 - 5 = 24.6334$

Take $x_1 = 3.1429$, 2nd approximation

$x_2 = 3.1429 - \frac{9.3305}{24.6334} = 3.1429 - 0.3788$

$x_2 = 2.7641$

$f(x_2) = f(2.7641) = 21.1184 - 13.8205 - 6 = 1.2979$

$f'(x_2) = f'(2.7641) = 22.9207 - 5 = 17.9207$

Take $x_2 = 2.7641$, 3rd approximation,

$x_3 = 2.7641 - \frac{1.2979}{17.9207} = 2.7641 - 0.0724$

$x_3 = 2.6917$

$f(x_3) = f(2.6917) = 19.5020 - 13.4585 - 6$

$f(x_3) = 0.0435$

$f'(x_3) = 21.7356 - 5 = 16.7356$

Take $x_3 = 2.6917$, fourth approximation,

$$x_4 = 2.6917 - \frac{0.0435}{16.7356} = 2.6917 - 0.0025$$

$$x_4 = 2.6892$$

$f(x_4) = 19.4477 - 13.446 - 6 = 0.0013$

$f'(x_4) = 21.6954 - 5 = 16.6954$

Take $x_4 = 2.6892$, 5^{th} approximation

$$x_5 = 2.6892 - \frac{0.0013}{16.6954} = 2.6892 - 0.0000$$

$$x_5 = 2.6892$$

Here $x_4 = x_5$

Hence, the required root is 2.6892.

12) Find the approximate value of $\sqrt{5}$ using Newton Raphson method.

Let $x = \sqrt{5} \Rightarrow x^2 = 5$

$x^2 - 5 = 0$

$f(x) = x^2 - 5$

$f'(x) = 2x$.

$f(0) = 0^2 - 5 = -5$

$f(1) = 1^2 - 5 = -4$

$f(2) = 4 - 5 = -1 < 0$

$f(3) = 9 - 5 = 4 > 0$

Hence the root lies between 2 & 3.

Take $x_0 = 2$, 1st approximation

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$f(2) = f(x_0) = 2^2 - 5 = -1$

$f'(2) = 2(2) = 4.$

$$x_1 = 2 + \frac{1}{4} = 2 + 0.25 = 2.25$$

2nd approximation:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$f(x_1) = (2.25)^2 - 5 = 0.0625$

$f'(x_1) = 2(2.25) = 4.5$

$$x_2 = 2.25 - \frac{0.0625}{4.5} = 2.25 - 0.0139$$

$x_2 = 2.2361$

3rd approximation,

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$f(x_2) = (2.2361)^2 - 5 = 0.0001$

$f'(x_2) = 2(2.2361) = 4.4722$

$$x_3 = 2.2361 - \frac{0.0001}{4.4722} = 2.2361 - 0$$

$x_3 = 2.2361$

Here $x_2 = x_3$

Hence the required root is 2.2361

17) Find positive value of $\sqrt[3]{17}$ using Raphson method.

Let $x = \sqrt[3]{17} \Rightarrow x = (17)^{1/3}$

$x^3 = 17 \Rightarrow x^3 - 17 = 0$

$f(x) = x^3 - 17 \Rightarrow f'(x) = 3x^2$

$f(0) = 0 - 17 = -17 < 0$

$f(1) = 1 - 17 = -16 < 0$

$f(2) = 8 - 17 = -9 < 0$

$f(3) = 27 - 17 = 10 > 0$

One root of $f(x)=0$ lies between 2 & 3.

Take $x_0 = 2$,

$f'(x_0) = f'(2) = 3(4) = 12$

1st approximation, $x_1 = 2 - \frac{(-9)}{12} = 2 + \frac{9}{12}$

$x_1 = 2 + 0.75 = 2.75$

$f(x_1) = (2.75)^3 - 17 = 20.7969 - 17 = 3.7969$

$f'(x_1) = 3(2.75)^2 = 3(7.5625) = 22.6875$

Take $x_1 = 2.75$, 2nd approximation,

$x_2 = 2.75 - \frac{3.7969}{22.6875} = 2.75 - 0.1674$

$x_2 = 2.5826$

$f(x_2) = (2.5826)^3 - 17 = 17.2255 - 17$

$f(x_2) = 0.2255$

$f'(x_2) = 3(2.5826)^2 = 3(6.6698) = 20.0094$

Take $x_2 = 2.5826$, 3rd approximation,

$x_3 = 2.5826 - \frac{0.2255}{20.0094}$

$= 2.5826 - 0.0113$

~~$x_3 = 2.5657$~~ $x_3 = 2.5713$.

$f(x_3) = (2.5657)^3 - 17 = 16.8895 - 17 = -0.1105$

$f'(x_3) = 3(2.5657)^2 = 3(6.5828) = 19.7484$

Take $x_3 = 2.5657$, 4^{th} approximation.

$x_4 = 2.5657 + \frac{0.1105}{19.7484} = 2.5657 + 0.0056$

$x_4 = 2.5713$

$f(x_3) = (2.5713)^3 - 17 = 17.0004 - 17 = 0.0004$

$f'(x_3) = 3(2.5713)^2 = 3(6.6116) = 19.8348$

Take $x_3 = 2.5713$, 5^{th} approximation

$x_4 = 2.5713 - \frac{0.0004}{19.8348} = 2.5713 - 0.0000$

$x_4 = 2.5713$.

Here $x_3 = x_4$,

Hence the required root is 2.5713.

---

3) $2x^3 - 3x - 6 = 0$

$f(x) = 2x^3 - 3x - 6$

$f'(x) = 6x^2 - 3$

$f(0) = 0 - 0 - 6 = -6$

$f(1) = 2 - 3 - 6 = -7$

$f(2) = 16 - 6 - 6 = 4$

one root lies between 1 & 2.

Take $x_0 = 1$, $f'(x_0) = f'(1) = 6 - 3 = 3$, $f(1) = -7$

1^{st} approximation, $x_1 = 1 + \frac{7}{3} = 1 + 2.3333$

$x_1 = 3.3333$

$f(x_1) = f(3.3333) = 2(3.3333)^3 - 3(3.3333) - 6$

$f(x_1) = 58.0702$.

$f'(x_1) = f'(3.3333) = 6(3.3333)^2 - 3 = 63.6653$

Take $x_1 = 3.3333$, 2^{nd} approximation,

$$x_2 = 3.3333 - \frac{59.0702}{63.6653}$$

$$x_2 = 2.4212$$

$$f(x_2) = f(2.4212) = 2(2.4212)^3 - 3(2.4212) - 6$$

$$f(x_2) = 18.1236.$$

$$f'(x_2) = f'(2.4212) = 6(2.4212)^2 - 3 = 29.1733.$$

Take $x_2 = 2.4212$, 3^{rd} approximation,

$$x_3 = 2.4212 - \frac{18.1236}{29.1733}$$

$$x_3 = 1.9028$$

$$f(x_3) = f(1.9028) = 2(1.9028)^3 - 3(1.9028) - 6$$

$$f(x_3) = 2.0703.$$

$$f'(x_3) = f'(1.9028) = 6(1.9028)^2 - 3 = 18.7239$$

Take $x_3 = 1.9028$, 4^{th} approximation,

$$x_4 = 1.9028 - \frac{2.0703}{18.7239} = 1.7922$$

$$f(x_4) = f(1.7922) = 2(1.7922)^3 - 3(1.7922) - 6$$

$$f(x_4) = 0.1364$$

$$f'(x_4) = f'(1.7922) = 6(1.7922)^2 - 3 = 16.2719$$

Take $x_4 = 1.7922$, 5^{th} approximation,

$$x_5 = 1.7922 - \frac{0.1364}{16.2719} = 1.7838$$

$f(x_5) = f(1.7838) = 2(1.7838)^3 - 3(1.7838) - 6$

$f(x_5) = 0.0005$

$f'(x_5) = f'(1.7838) = 6(1.7838)^2 - 3 = 16.0917$

Take $x_5 = 1.7838$, 6^{th} approximation.

$$x_6 = 1.7838 - \frac{0.0005}{16.0917}$$

$x_6 = 1.7838$

Here. $x_5 = x_6$

The required root is 1.7838

---

8) $x \cos x = 0$

$f(x) = x \cos x$

$f(0) = 0(\cos 0) = 0$

$f(\pi/2) = \pi/2 \cos \pi/2 = 0$

$f(\pi) = \pi \cos \pi = -3.1416$

$\therefore$ The root lies between $\pi/2$ and π

Take $x_0 = \pi/2$ $\qquad f'(x) = -x \sin x + \cos x$

$f'(x_0) = -\frac{\pi}{2} \sin \pi/2 = -1.5708$

1^{st} approximation, $x_1 = \pi/2 - \frac{0}{(-1.5708)} = \pi/2 = 1.5708$

$f(x_1) = f(1.5708) = (1.5708) \cos 1.5708$

$= 1.5708(0.0000)$

$f(x_1) = 0$

Take $x_1 = 1.5708$, 2^{nd} approximation,

$f'(x_1) = f'(1.5708) = -\sin(1.5708) = -1$

$$x_2 = 1.5708 - \frac{0}{(-1.5708)} = 1.5708$$

Here, $x_1 = x_2$

$\therefore$ The required root is 1.5708

16) Find the negative root of $x^3-5x+11=0$ by using Newton Raphson method correct to 2 places of decimals.

Soln:

Let $g(x) = x^3-5x+11$. We find (+ve) root of $g(-x)=0$.

(ie) $g(-x) = (-x)^3-5(-x)+11 = -x^3+5x+11 = 0$.

Hence, we have to find the (+ve) root of

$f(x) = x^3-5x-11 = 0$

$f(0) = 0-0-11 = -11$

$f(1) = 1-5-11 = -14$

$f(2) = 8-10-11 = -13$

$f(3) = 27-15-11 = 1$

$\therefore$ The root of $f(x)=0$ lies between 2 and 3.

$f'(x) = 3x^2-5$

Take $x_0 = 2$, 1st approximation

$f'(x_0) = f'(2) = 3(4)-5 = 7$

$$x_1 = 2 - \frac{-13}{7} = \frac{14+13}{7} = \frac{27}{7} = 3.8571$$

$f(x_1) = f(3.8571) = (3.8571)^3 - 5(3.8571) - 11$

$= 57.3829 - 19.2855 - 11$

$f(x_1) = 27.0974$

$f'(x_1) = 3(3.8571)^2 - 5 = 44.6316 - 5$

$f'(x_1) = 39.6316$.

Take $x_\phi = 3.8571$, 2nd approximation

$$x_2 = 3.8571 - \frac{27.0974}{39.6316} = 3.8571 - 0.6837$$

$x_2 = 3.1734$

$f(x_2) = (3.1734)^3 - 5(3.1734) - 11 = 31.9576 - 15.8670 - 11$

$f(x_2) = 5.0906$

$f'(x_2) = 3(3.1734)^2 - 5 = 30.2115 - 5 = 25.2115$

Take $x_2 = 3.1734$, 3rd approximation.

$$x_3 = 3.1734 - \frac{5.0906}{25.2115} = 3.1734 - 0.2019$$

$x_3 = 2.9715$

$f(x_3) = (2.9715)^3 - 5(2.9715) - 11 = 26.2378 - 14.8575 - 11$

$f(x_3) = 0.3803$

$f'(x_3) = 3(2.9715)^2 - 5 = 26.4894 - 5$

$= 21.4894$

Take $x_3 = 2.9715$, 4th approximation

$$x_4 = 2.9715 - \frac{0.3803}{21.4894} = 2.9715 - 0.0177$$

$x_4 = 2.9538$

$f(x_4) = (2.9538)^3 - 5(2.9538) - 11 = 25.7717 - 14.7690$

$f(x_4) = 0.00$

$f'(x_4) = 3(2.9538)^2 - 5 = 26.1747 - 5 = 21.1747$

Take $x_4 = 2.9538$, 5th approximation

$$x_5 = 2.9538 - \frac{0.0020}{21.1747} = 2.9538 - 0.0000$$

$x_5 = 2.9538$

Here $x_4 = x_5 = 2.9538$ (correct to four decimals.

We take $f(x) = x^3 - 2x - 5 = 0$ to be 2.9538.

Hence the negative root of $x^3-5x+11=0$ is

-2.9538

Hence the required root of $g(x)$ is -2.9538

17) State true or false.

i) finding a (-ve) root of $2x^3-3x+6=0$ is equivalent to finding a positive root of $2x^3-3x-6=0$

Ans : True

ii) finding a (-ve) root of $x^3-6x-4=0$ is equivalent to finding a positive root of $-x^3+6x+4=0$

Ans : false [It is equivalent to find a (+ve) root of x^3-6x+4]

iii) finding a (-ve) root of $x^4+x=10$ is equivalent to finding a (+ve) root of $x^4+x-10=0$.

Ans : false [$x^4+x=10$ is equivalent to $x^4+x-10=0$]

Unit 2

Unit – II

Interpolation : Finite differences – Forward differences, Backward differences, Central differences, Symbolic relations, Newton's formula for Interpolation – Interpolation with unevenly spaced points – Lagrange's Interpolation formula.

Unit - II

Finite Differences

Difference operators:

* forward
* backward and
* central difference operators

Forward difference operator:

$$\Delta[f(x)] = f(x+h) - f(x)$$

Backward difference operator:

$$\nabla[f(x)] = f(x) - f(x-h)$$

central difference operator:

$$\delta(f(x)) = f(x+h/2) - f(x-h/2)$$

Factorial function:

$$x^{(n)} = x(x-h)(x-2h)\dots\dots[x-(n-1)h]$$

Pbm: 1

Form the forward diff. table for the following. data

x	0	1	2	3	4
y	8	11	9	15	6

Soln:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	8				
		3			
1	11		-5		
		-2		13	
2	9		8		-36
		6		-23	
3	15		-15		
		-9			
4	6				

Pbm: 2 Find $\Delta(2^x)$

Soln:

$$\Delta[f(x)] = f(x+h) - f(x)$$

$$\Delta(2^x) = 2^{x+h} - 2^x$$

$$\Delta(2^x) = 2^x(2^h - 1)$$

Pbm: 3 . find the n^{th} diff. of e^x.

Soln:

$$\Delta(e^x) = e^{x+h} - e^x = e^x(e^h - 1)$$

$$\Delta^2(e^x) = \Delta[\Delta(e^x)]$$

$$= \Delta[e^x(e^h - 1)]$$

$$= (e^h - 1)\Delta(e^x)$$

$$= (e^h - 1)(e^{x+h} - e^x)$$

$$= (e^h - 1)(e^x)(e^h - 1)$$

$$\Delta^2(e^x) = e^x \cdot (e^h - 1)^2$$

$$\Delta^3(e^x) = \Delta(\Delta^2(e^x))$$

$$= \Delta(e^x(e^h - 1)^2)$$

$$= (e^h - 1)^2 \Delta(e^x)$$

$$= (e^h - 1)^2 (e^{x+h} - e^x)$$

$$= (e^h - 1)^2 (e^h - 1)(e^x)$$

$$\Delta^3(e^x) = e^x(e^h - 1)^3$$

Proceeding like this, we get the n^{th} difference as

$$\Delta^n(e^x) = e^x(e^h - 1)^n$$

Pbm : 4.

P.T $\Delta \log f(x) = \log\left[1+\frac{\Delta f(x)}{f(x)}\right]$

Soln:

$$\Delta \log f(x) = \log f(x+h) - \log f(x)$$

$$= \log\left[\frac{f(x+h)}{f(x)}\right]$$

$$= \log\left[\frac{f(x+h)+f(x)-f(x)}{f(x)}\right]$$

$$= \log\left[\frac{f(x)+\Delta f(x)}{f(x)}\right]$$

$$= \log\left[1+\frac{\Delta f(x)}{f(x)}\right]$$

Pbm : 5

P.T $\Delta \tan^{-1}\left[\left(\frac{n-1}{n}\right)\right] = \tan^{-1}\left(\frac{1}{2n^2}\right)$

Soln:

without loss of generality we take the interval of differencing as $h=1$

$$\Delta \tan^{-1}\left[\left(\frac{n-1}{n}\right)\right] = \tan^{-1}\left(\frac{(n+1)-1}{n+1}\right) - \tan^{-1}\left(\frac{n-1}{n}\right)$$

(by defn)

$\tan^{-1}A - \tan^{-1}B = \tan^{-1}\left(\frac{A-B}{1+AB}\right)$

$$= \tan^{-1}\left(\frac{n}{n+1}\right) - \tan^{-1}\left(\frac{n-1}{n}\right)$$

$$= \tan^{-1}\left(\frac{\frac{n}{n+1} - \frac{n-1}{n}}{1+\frac{n}{n+1}\times\frac{n-1}{n}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{n^2-(n+1)(n-1)}{n(n+1)}}{\frac{(n+1)n+n(n-1)}{n(n+1)}}\right)$$

$$= \tan^{-1}\left(\frac{n^2-(n^2-1)}{n(n+1)+n(n-1)}\right)$$

$$= \tan^{-1}\left(\frac{1}{n^2+1+n^2-1}\right)$$

$$\Delta \tan^{-1}\left[\left(\frac{n-1}{n}\right)\right] = \tan^{-1}\left(\frac{1}{2n^2}\right)$$

Hence proved.

Pbm: 6

If $f(x) = \dfrac{x}{x^2+7x+12}$ find $\Delta f(x)$ taking the interval of differencing as unity.

$\downarrow$

$h = 1$

soln:

$$f(x) = \frac{x}{x^2+7x+12} = \frac{A}{x+3} - \frac{B}{x+3}.$$

By partial fraction

$$x = A(x+3) + B(x+4)$$

$x = -4$,

$$-4 = A(-1) + B(0)$$

$$A = 4$$

$x = -3$,

$$-3 = A(0) + B(-3+4)$$

$$B = -3$$

$$f(x) = \frac{x}{x^2+7x+12} = \frac{A}{x+4} - \frac{B}{x+3}$$

$$f(x) = \frac{4}{x+4} - \frac{3}{x+3}$$

$[\Delta f(x) = f(x+h) - f(x)]$

$\therefore h = 1$

$$\Delta f(x) = \left[\frac{4}{(x+1)+4} - \frac{3}{(x+1)+3}\right] - \left[\frac{4}{x+4} - \frac{3}{x+3}\right]$$

$$= \frac{4}{x+5} - \frac{3}{x+4} - \frac{4}{x+4} + \frac{3}{x+3}$$

Pbm: 7

find th

$f(x) = ab^{cx}$

soln:

Pbm: 8

Ev

of diffe

soln:

$$= \frac{1}{x+5} - \frac{7}{x+4} + \frac{2}{x+3}$$

Pbm: 7

Find the 1st & 2nd order differences for $f(x) = ab^{cx}$

Soln: $\Delta f(x) = f(x+h) - f(x)$

$$= ab^{c(x+h)} - ab^{cx}$$

$$= ab^{cx} b^{ch} - ab^{cx}$$

$$\Delta f(x) = ab^{cx}(b^{ch} - 1)$$

$$\Delta^2 f(x) = \Delta(\Delta f(x))$$

$$= \Delta [ab^{cx}(b^{ch}-1)]$$

$$= (b^{ch}-1)\,\Delta(ab^{cx})$$

$$= (b^{ch}-1)[ab^{c(x+h)} - ab^{cx}]$$

$$= (b^{ch}-1)(ab^{cx} b^{ch} - ab^{cx})$$

$$= (b^{ch}-1)(ab^{cx})(b^{ch}-1)$$

$$\Delta^2 f(x) = (ab^{cx})(b^{ch}-1)^2$$

Pbm: 8

Evaluate $(\Delta - \nabla)x^2$ taking the interval of differencing as h.

Soln: $(\Delta - \nabla)x^2 = \Delta x^2 - \nabla x^2$

$$\Delta f(x) = f(x+h) - f(x)$$

$$\nabla f(x) = f(x) - f(x-h)$$

$$(\Delta - \nabla)x^2 = \Delta x^2 - \nabla x^2$$

$$= (x+h)^2 - x^2 - [x^2 - (x-h)^2]$$

$$= x^2 + h^2 + 2xh - x^2 - [x^2 - x^2 - h^2 + 2xh]$$

$$= (2xh + h^2) - (2xh - h^2)$$

$$(\Delta - \nabla)x^2 = 2h^2$$

Pbm: 9 - Find $\Delta^n \sin x$ taking $h=1$

Soln:

$\Delta \sin x = \sin(x+1) - \sin x$ $\qquad \sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

$= 2\cos(x+1/2)\sin(1/2)$ $\qquad \rightarrow \sin(90^\circ+\theta) = \cos\theta$

$\Delta \sin x = 2\sin(1/2)\sin(\pi/2 + x + 1/2)$

$\Delta^2 \sin x = 2\sin(1/2)\,\Delta \sin(\pi/2 + x + 1/2)$

$= 2\sin(1/2)\left[\sin(\pi/2 + x + 1 + 1/2) - \sin(\pi/2 + x + 1/2)\right]$

$= 2\sin(1/2)\left[2\cos(\pi/2 + x + 1)\sin(1/2)\right]$

$= \left[2\sin(1/2)\right]^2 \sin\left[\pi/2 + (\pi/2 + x + 1)\right]$

$\Delta^2 \sin x = \left[2\sin(1/2)\right]^2 \sin\left[2(\pi/2 + 1/2) + x\right]$

Proceeding like this,

$\Delta^n \sin x = \left[2\sin(1/2)\right]^n \sin\left[n(\pi/2 + 1/2) + x\right]$

Pbm: 10

Evaluate $\Delta^{10}[(1-x)(1-2x^2)(1-3x^3)(1-4x^4)]$

Soln:

Let $y = [(1-x)(1-2x^2)(1-3x^3)(1-4x^4)]$

This is a polynomial of degree 10 with leading coefficients 24.

Since $\Delta^n y = \begin{cases} 0 & \text{if } r > 10 \\ \text{constant} & \text{if } r = 10 \end{cases}$

$\therefore$ deg $= 1+2+3+4$
deg $= 10$
$\therefore$ coefficients $= 1\times2\times3\times4$
coeff $= 24$

we have

$(h=1)$

$\Delta^{10} y = \Delta^{10}(24x^{10})$

$= 24\,\Delta^{10}(x^{10})$

$= 24 \times 10!$

Pbm: 11

Find the

Soln: Let

Differen

below

since

x	y
0	2
1	6
2	12
3	20
4	30
5	a

Pbm: 12

If

Let

x	
1	
2	
3	
4	
5	

Pbm : 11

Find the 6^{th} term of the sequence 2, 6, 12, 20, 30, ...

soln: Let a be the 6^{th} term.

Difference table for the given data is given below.

since five values are given $\Delta^5 y = 0$

~~Hence~~ $\cancel{\alpha - 42 = 0}$

$\cancel{\alpha = 42}$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	2					
		4				
1	6		2			
		6		0		
2	12				0	
			2			$a-42$
3	20	8		0		
4	30	10	2		$a-42$	
				$a-42$		
5	a	$a-30$	$a-40$			

Hence $a - 42 = 0$

$a = 42$

$\therefore$ 6^{th} term of the given sequence is 42.

Pbm : 12.

If $u_1 = 1$, $u_3 = 17$, $u_4 = 43$, $u_5 = 89$, find u_2.

Let $u_2 = a$, we have the diff. table.

x	u	Δu	$\Delta^2 u$	$\Delta^3 u$	$\Delta^4 u$
1	1				
		$a-1$			
2	a		$18-2a$		
		$17-a$		$-9+3a$	
3	17				$20-4a$
			$9+a$		
		26			
4	43		20	$11-a$	
5	89	46			

Hence, four values are given $\Delta^4 u = 0$

$$20 - 4a = 0$$
$$20 = 4a$$
$$a = 5.$$

Hence, $u_2 = 5$

Pbm: 13

Express $2x^3 - 3x^2 + 4x - 8$ as a factorial polynomial

Soln: Let the factorial polynomial be

$$Ax^{(3)} + Bx^{(2)} + Cx^{(1)} + D$$

$$2x^3 - 3x^2 + 4x - 8 = Ax^{(3)} + Bx^{(2)} + Cx^{(1)} + D$$

where A, B, C, D are to be determined by synthetic division

0	2	-3	4	-8
	0	0	0	0
1	2	-3	4	$\boxed{-8}$
	0	2	-1	
2	2	-1	$\boxed{3}$	
	0	4		
	2	$\boxed{3}$		

$\therefore$ The factorial polynomial is $2x^{(3)} + 3x^{(2)} + 3x^{(1)} - 8$.

Pbm. 14

ST $\Delta(5x^4 + 6x^3 + x^2 - x + 7) = 20x^{(3)} + 108x^{(2)} + 108x^{(1)} + 11$

Soln:

Let $y = 5x^4 + 6x^3 + x^2 - x + 7$

Let $Ax^{(4)} + Bx^{(3)} + Cx^{(2)} + Dx^{(1)} + E$ be the factorial polynomial of y.

$$\begin{array}{c|ccccc}
0 & 5 & 6 & 1 & -1 & 7 \\
 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & 5 & 6 & 1 & -1 & \boxed{7} \\
 & 0 & 5 & 11 & 12 & \\
\hline
2 & 5 & 11 & 12 & \boxed{11} & \\
 & 0 & 10 & 42 & & \\
\hline
3 & 5 & 21 & \boxed{54} & & \\
 & 0 & 15 & & & \\
\hline
 & 5 & \boxed{36} & & &
\end{array}$$

$y = 5x^{(4)} + 36x^{(3)} + 54x^{(2)} + 11x^{(1)} + 7$

$\Delta y = 20x^{(3)} + 108x^{(2)} + 108x^{(1)} + 11$

Pbm: 15

Find the 2^{nd} diff. of the polynomial

$f(x) = x^4 - 12x^3 + 42x^2 - 30x + 9$ with $h = 2$.

Soln:

First we shall express the given polynomial $f(x)$ in terms of the factorial polynomial by synthetic division with $h = 2$

$$\begin{array}{c|ccccc}
0 & 1 & -12 & 42 & -30 & 9 \\
 & 0 & 0 & 0 & 0 & 0 \\
\hline
2 & 1 & -12 & 42 & -30 & \boxed{9} \\
 & 0 & 2 & -20 & 44 & \\
\hline
 & 1 & -10 & 22 & \boxed{14} & \\
4 & 0 & 4 & -24 & & \\
\hline
6 & 1 & -6 & \boxed{-2} & & \\
 & 0 & 6 & & & \\
\hline
 & 1 & \boxed{0} & & &
\end{array}$$

$f(x) = x^{(4)} - 2x^{(2)} + 14x^{(1)} + 9$

$\Delta f(x) = 4x^{(3)} - 4x^{(1)} + 14 \quad (h=2)$

$= 8x^{(3)} - 8x^{(1)} + 28$

$\Delta^2 f(x) = 24x^{(2)} - 8$

$= 48x^{(2)} - 16$

$= 48x(x-2) - 16$

$= 48x^2 - 96x - 16$

Pbm: 16

Find the function whose first diff is $x^3+3x^2+5x+12$.

Soln:

Given $\Delta y = x^3 + 3x^2 + 5x + 12$

we express this in terms of factorial polynomial with $h=1$

0	1	3	5	12
	0	0	0	0
	1	3	5	$\boxed{12}$
1	0	1	4	
	1	4	$\boxed{9}$	
2	0	2		
	1	$\boxed{6}$		

$\Delta y = x^{(3)} + 6x^{(2)} + 9x^{(1)} + 12$

$y = \Delta^{-1}\left[x^{(3)} + 6x^{(2)} + 9x^{(1)} + 12\right]$

$= \frac{x^{(4)}}{4} + \frac{6x^{(3)}}{3} + \frac{9x^{(2)}}{2} + \frac{12x^{(1)}}{1} + c$

$= \frac{x^{(4)}}{4} + 2x^{(3)} + \frac{9x^{(2)}}{2} + 12x^{(1)} + c$

$= \frac{1}{4}\left[x(x-1)(x-2)(x-3)\right] + 2\left[x(x-1)(x-2)\right]$

$+ \frac{9}{2}\left[x(x-1)\right] + 12x + c$

Pbm: 17

If $y = \dfrac{1}{(3x+1)(3x+4)(3x+7)}$ S.T

$$\Delta^2 y = \frac{108}{(3x+1)(3x+4)(3x+7)(3x+10)(3x+13)}$$

Soln:

$$y = \frac{1}{(3x+1)(3x+4)(3x+7)}$$

$$= \frac{1}{3^3(x+1/3)(x+4/3)(x+7/3)}$$

$$= \frac{1}{27(x-2/3+1)(x-2/3+2)(x-2/3+3)}$$

$$= \frac{1}{27(x-2/3)^{(3)}}$$

$$y = \frac{1}{27}(x-2/3)^{-3}$$

$$\Delta y = \frac{1}{27}(-3)(x-2/3)^{(-4)}$$

$$\Delta^2 y = \frac{1}{27}(-3)(-4)(x-2/3)^{(-5)}$$

$$= \frac{12}{27[(x-2/3)+1][(x-2/3)+2][(x-2/3)+3][(x-2/3)+4][(x-2/3)+5]}$$

$$= \frac{12}{27\left[\frac{3x-2+3}{3}\right]\left[\frac{3x-2+6}{3}\right]\left[\frac{3x-2+9}{3}\right]\left[\frac{3x-2+12}{3}\right]\left[\frac{3x-2+15}{3}\right]}$$

$$= \frac{12\cdot 3^5}{27\cdot(3x+1)(3x+4)(3x+7)(3x+10)(3x+13)}$$

$$= \frac{12 \times 9}{(3x+1)(3x+4)(3x+7)(3x+10)(3x+13)}$$

$$\Delta^2 y = \frac{108}{(3x+1)(3x+4)(3x+7)(3x+10)(3x+13)}$$

Hence proved.

Unit - 2.
Finite Differences

P. Shanmugapriya

3.2 Other Difference Operators

i) Shift operator

$\Delta f(x) = f(x+h) - f(x)$

$\nabla f(x) = f(x) - f(x-h)$

$E f(x) = f(x+h)$

$E^{-1} f(x) = f(x-h)$

$$E f(x) = f(x+h)$$

ii) Inverse operator

$$E^{-1} f(x) = f(x-h)$$

iii) Averaging operator

$$\mu f(x) = \frac{f(x+h/2) + f(x-h/2)}{2}$$

iv) Relation between E and Δ

$$E = 1 + \Delta$$

v) Relation between E^{-1} and ∇

$$\nabla = 1 - E^{-1}$$

vi) $\delta = E^{1/2} - E^{-1/2}$

vii) $\mu = \dfrac{E^{1/2} + E^{-1/2}}{2}$

viii) $\delta = E^{1/2} \nabla$

ix) $E = e^{hD}$

x) $D = \frac{1}{h}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \cdots\right]$

Pbm 1 $\quad E\nabla = \nabla E = \Delta$

soln:

$$E\nabla = E(1-E^{-1})$$

$$= E - E^{-1}E \qquad \nabla = 1 - E^{-1}$$

$$= E - 1$$

$$E\nabla = \Delta \quad \rightarrow (1)$$

$$\nabla E = (1-E^{-1})E$$

$$= E - E^{-1}E$$

$$\nabla E = E - 1$$

$$\nabla E = \Delta \quad \rightarrow (2)$$

From (1) & (2),

$$E\nabla = \nabla E = \Delta.$$

Hence proved.

Pbm 2

$$(E^{1/2} + E^{-1/2})(1+\Delta)^{1/2} = 2 + \Delta$$

soln:

$$(E^{1/2} + E^{-1/2})(1+\Delta)^{1/2} = (E^{1/2} + E^{-1/2})E^{1/2}$$

$E = 1+\Delta$

$$= E^{1/2}\cdot E^{1/2} + E^{-1/2}\cdot E^{1/2}$$

$$= E^{1/2+1/2} + E^{-1/2+1/2}$$

$$= E^{1} + E^{0}$$

$$= E + 1$$

$$= (1+\Delta) + 1$$

$$= 1 + \Delta + 1$$

$$= 2 + \Delta$$

Hence proved.

3) $\Delta\nabla = \Delta$

soln:

Also,

From

4) Pro

soln:

3) $\Delta\nabla = \Delta - \nabla = \delta^2$

Soln:

$\Delta\nabla = (E-1)(1-E^{-1}) = E - EE^{-1} - 1 + E^{-1}$

$= E - E^{1-1} - 1 + E^{-1}$

$= E - E^{0} - 1 + E^{-1}$

$= E - 1 - 1 + E^{-1}$

$= E + E^{-1} - 2$

$= (E^{1/2})^2 + (E^{-1/2})^2 - 2E^{1/2}E^{-1/2}$

$= (E^{1/2} - E^{-1/2})^2$

$\Delta\nabla = \delta^2 \rightarrow ①$

Also, $\Delta - \nabla = (E-1) - (1-E^{-1})$

$= E - 1 - 1 + E^{-1}$

$= E + E^{-1} - 2$

$= (E^{1/2})^2 + (E^{-1/2})^2 - 2E^{1/2}E^{-1/2}$

$= (E^{1/2} - E^{-1/2})^2$

$\Delta - \nabla = \delta^2 \rightarrow ②$

From ① & ②, $\Delta\nabla = \Delta - \nabla = \delta^2$.

4) Prove that $E^{1/2} = \mu + \frac{1}{2}\delta$

Soln:

W.K.T, $\mu = \dfrac{E^{1/2} + E^{-1/2}}{2}$ and

$\delta = (E^{1/2} - E^{-1/2})$

RHS,

$\mu + \frac{1}{2}\delta = \dfrac{E^{1/2} + E^{-1/2}}{2} + \dfrac{E^{1/2} - E^{-1/2}}{2}$

$= \dfrac{E^{1/2} + E^{-1/2} + E^{1/2} - E^{-1/2}}{2}$

$$= \frac{2E^{1/2}}{2} = E^{1/2} = L-HS$$

$$\mu + \frac{1}{2}\delta = E^{1/2}$$

Hence proved.

Pbm : 5 $\quad \mu\delta = \frac{\Delta}{2} + \frac{\Delta E^{-1}}{2}$

soln:

$$\frac{\Delta}{2} + \frac{\Delta E^{-1}}{2} = \frac{\Delta}{2}(1+E^{-1})$$

$$= \frac{1}{2}(E-1)(1+E^{-1})$$

$$= \frac{1}{2}(E + EE^{-1} - 1 - E^{-1})$$

$$= \frac{1}{2}(E + 1 - 1 - E^{-1})$$

$$= \frac{1}{2}(E - E^{-1})$$

$$= \frac{1}{2}\left[(E^{1/2})^2 - (E^{-1/2})^2\right]$$

$$= \frac{1}{2}\left[(E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2})\right]$$

$$= \frac{E^{1/2} + E^{-1/2}}{2}(E^{1/2} - E^{-1/2})$$

$\frac{\Delta}{2} + \frac{\Delta E^{-1}}{2} = \mu\delta$ (By definition of μ and δ)

Hence proved.

Pbm : 6 $\quad$ P.T $\quad 1 - e^{-hD} = \nabla$

soln:

W.K.T

$$D = \frac{1}{h}\log E$$

$$hD = \log E$$

$$e^{hD} = E$$

$$\frac{1}{e^{hD}} = \frac{1}{E}$$

$$e^{-hD} = E^{-1}$$

$$= 1-\nabla$$

$$1-e^{-hD} = \nabla$$

Hence proved.

Pbm : 7

P.T $\Delta + \nabla = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$

soln:

RHS,

$$\frac{\Delta}{\nabla} - \frac{\nabla}{\Delta} = \frac{E-1}{1-E^{-1}} - \frac{1-E^{-1}}{E-1}$$

$$= \frac{(E-1)^2 - (1-E^{-1})^2}{(1-E^{-1})(E-1)}$$

$$= \frac{(E^2+1-2E) - (1+E^{-2}-2E^{-1})}{E-1-E^{-1}E+E^{-1}}$$

$$= \frac{E^2+1-2E-1-E^{-2}+2E^{-1}}{E-2+E^{-1}}$$

$$= \frac{\quad}{E+E^{-1}-2}$$

$$= -\frac{2(E+E^{-1})}{E+E^{-1}} - \frac{2(E+E^{-1})}{(-2)}$$

$$= -2 + E+E^{-1}$$

Pbm: 11 Taking $h=1$ find $(\Delta+\nabla)^2 f(x)$ where $f(x) = x^2+x$

Soln:

$$(\Delta+\nabla)^2 f(x) = (E-1+1-E^{-1})^2 (x^2+x)$$

$$= (E-E^{-1})^2 (x^2+x)$$

$$= (E^2+E^{-2}-2)(x^2+x)$$

$$= E^2x^2+E^2x+E^{-2}x^2+E^{-2}x-2x^2-2x$$

$$= (x+2)^2+(x+2)+(x-2)^2+(x-2)-2(x^2+x)$$

$$= x^2+4x+4+x+\cancel{2}+x^2-4x+4+x-\cancel{2}-2(x^2+x)$$

$$= 2(x^2+x)+8-2(x^2+x)$$

$$= 8$$

Pbm: 13 P.T $\Delta^2 y_2 = \nabla^2 y_4$

Soln:

$$\Delta^2 y_2 = (E-1)^2 y_2$$

$$= (E^2-2E+1)y_2 = E^2y_2-2Ey_2+y_2$$

$$\Delta^2 y_2 = y_4 - 2y_3 + y_2 \rightarrow ①$$

$$\nabla^2 y_4 = (1-E^{-1})^2 y_4$$

$$= (1+E^{-2}-2E^{-1}) y_4$$

$$= y_4 + E^{-2}y_4 - 2E^{-1}y_4$$

$$= y_4 + y_2 - 2y_3$$

$$\nabla^2 y_4 = y_4 - 2y_3 + y_2 \rightarrow ②$$

From ① & ②, we get

$$\Delta^2 y_2 = \nabla^2 y_4$$

Hence proved.

Pbm: 8

P.T 6

soln:

$\delta = \Delta E^{-1/2}$

Pbm: 12

soln:

$y_3 + \Delta$

Pbm : 8

P.T $\delta = \Delta E^{-1/2}$ and hence P.T $E = \left(\frac{\Delta}{\delta}\right)^2$

soln:

$$\Delta E^{-1/2} f(x) = \Delta f(x - h/2)$$

$$= f(x - h/2 + h) - f(x - h/2)$$

$$= f\left(x - \frac{h - 2h}{2}\right) - f(x - h/2)$$

$$= f(x + h/2) - f(x - h/2)$$

$$\Delta E^{-1/2} f(x) = \delta f(x)$$

$$\Delta E^{-1/2} = \delta$$

$$E^{-1/2} = \frac{\delta}{\Delta}$$

$$E^{1/2} = \frac{\Delta}{\delta}$$

$$\therefore E = \left(\frac{\Delta}{\delta}\right)^2$$

Pbm : 12

P.T $y_4 = y_3 + \Delta y_2 + \Delta^2 y_1 + \Delta^3 y_1$

soln:

$$y_3 + \Delta y_2 + \Delta^2 y_1 + \Delta^3 y_1 = y_3 + (E-1) y_2 + (E-1)^2 y_1 + (E-1)^3 y_1$$

$$= y_3 + E y_2 - y_2 + (E^2 - 2E + 1) y_1 + (E^3 - 3E^2 + 3E - 1) y_1$$

$$= y_3 + E y_2 - y_2 + E^2 y_1 - 2E y_1 + y_1 + E^3 y_1 - 3E^2 y_1 + 3E y_1 - y_1$$

$$= y_3 + y_3 - y_2 + y_3 - 2y_2 + y_1 + y_4 - 3y_3 + 3y_2 - y_1$$

$$= 3y_3 - 3y_2 + y_1 + y_4 - 3y_3 + 3y_2 - y_1$$

$$= y_4$$

Hence proved.

Pbm. 14

Given $u_0 = 2$, $u_1 = 11$, $u_2 = 80$, $u_3 = 200$, $u_4 = 100$, $u_5 = 8$. find $\nabla^5 u_5$.

i) without constructing the diff. table.

ii) by constructing the diff. table.

soln:

i) We know that

$$\nabla = 1 - E^{-1}$$

$$\nabla^5 u_5 = (1-E^{-1})^5 u_5$$

$$(x+a)^n = x^n + nc_1 x^{n-1} a + nc_2 x^{n-2} a^2 + \cdots$$

$$= (1 - 5E^{-1} + 10E^{-2} - 10E^{-3} + 5E^{-4} - E^{-5}) u_5$$

$$= u_5 - 5E^{-1}(u_5) + 10E^{-2}(u_5) - 10E^{-3}(u_5) + 5E^{-4}(u_5) - E^{-5}(u_5)$$

$$= u_5 - 5u_4 + 10u_3 - 10u_2 + 5u_1 - u_0$$

$$= 8 - 5(100) + 10(200) - 10(80) + 5(11) - 2$$

$$= 8 - 500 + 2000 - 800 + 55 - 2$$

$$= 8 + 1500 - 800 + 53$$

$$= 8 + 700 + 53$$

$$\therefore \nabla^5 u_5 = 761$$

ii) $\nabla^n u(x) = \Delta^n (x-n)$. Hence $\nabla^5 u_5 = \Delta^5 u_0$.

We construct forward difference table.

$1-x+x^2-x^3+x^4$

x

0

1

2

3

4

5

Pbm: 15 -

find

x	u	Δu	$\Delta^2 u$	$\Delta^3 u$	$\Delta^4 u$	$\Delta^5 u$
0	2					
		9				
1	11		60			
		69		-9		
2	80		51		-262	
						761
		120		-271		
3	200		-220		499	
		-100		228		
4	100		8			
		-92				
5	8					

$\therefore \nabla^5 u_5 = \Delta^5 u_0 = 761$

Pbm: 15.

If $u_0 = 1,\ u_1 = 5,\ u_2 = 8,\ u_3 = 3,\ u_4 = 7,\ u_5 = 0$.

find $\Delta^5 u_0$.

<u>soln:</u> $\Delta^5 u_0 = (E-1)^5 u_0$

$= [E^5 + 5C_1 E^4 (-1) + 5C_2 E^3 (-1)^2 + 5C_3 E^2 (-1)^3$

$+ 5C_4 E (-1)^4 + 5C_5 (-1)^5] u_0$

$= (E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1) u_0$

$= E^5 u_0 - 5E^4 u_0 + 10E^3 u_0 - 10E^2 u_0 + 5E u_0 - u_0$

$= u_5 - 5u_4 + 10u_3 - 10u_2 + 5u_1 - u_0$

$= 0 - 5(7) + 10(3) - 10(8) + 5(5) - 1$

$= -35 + 30 - 80 + 25 - 1$

$= -5 - 80 + 24$

$\Delta^5 u_0 = -61$

Date :

P.T $\frac{1}{2}\delta^2 + \delta\sqrt{1+\frac{\delta^2}{4}} = \Delta$

Soln:

$$\frac{1}{2}\delta^2 + \delta\sqrt{1+\frac{\delta^2}{4}} = \frac{1}{2}\delta\left[\delta + 2\sqrt{1+\frac{\delta^2}{4}}\right]$$

$$= \frac{1}{2}\delta\left[\delta + \frac{2}{2}\sqrt{4+\delta^2}\right]$$

$$= \frac{1}{2}\delta\left[\delta + \sqrt{4+\delta^2}\right]$$

$$= \frac{1}{2}\delta\left[(E^{1/2} - E^{-1/2}) + \sqrt{4 + (E^{1/2} - E^{-1/2})^2}\right]$$

$$= \frac{1}{2}\delta\left[(E^{1/2} - E^{-1/2}) + \sqrt{4 + E + E^{-1} - 2}\right]$$

$$= \frac{1}{2}\delta\left[(E^{1/2} - E^{-1/2}) + \sqrt{E + E^{-1} + 2}\right]$$

$$= \frac{1}{2}\delta\left[(E^{1/2} - E^{-1/2}) + \sqrt{(E^{1/2} + E^{-1/2})^2}\right]$$

$$= \frac{1}{2}\delta\left[(E^{1/2} - E^{-1/2}) + (E^{1/2} + E^{-1/2})\right]$$

$$= \frac{1}{2}\delta\left[E^{1/2} - E^{-1/2} + E^{1/2} + E^{-1/2}\right]$$

$$= \frac{1}{2}\delta \cdot 2E^{1/2}$$

$$= \delta \cdot E^{1/2}$$

$$= (E^{1/2} - E^{-1/2})E^{1/2}$$

$$= E^{1/2 + 1/2} - E^{-1/2 + 1/2}$$

$$= E^1 - E^0$$

$$= E - 1$$

$$= \Delta$$

Date : 10

Prove

Integer

Soln: we

we

on r

Pbm : 10.

Prove $\nabla^r f(x) = \Delta^r f(x-r)$ for any positive integer.

Soln: We know that $\nabla = 1 - E^{-1}$ and $\Delta = E - 1$

we prove the required result by induction on r.

$$\nabla f(x) = (1-E^{-1}) f(x)$$
$$= f(x) - E^{-1} f(x)$$
$$= f(x) - f(x-1)$$
$$= \Delta f(x-1)$$

The result is true for $r = 1$

Let us assume that the result is true for $r = K$

$$\therefore \nabla^K f(x) = \Delta^K f(x-K)$$

Now, $\nabla^{K+1} f(x) = \nabla(\nabla^K f(x))$

$$= \nabla(\Delta^K f(x-K))$$
$$= [(1-E^{-1}) \Delta^K f(x-K)]$$
$$= \Delta^K f(x-K) - E^{-1} \Delta^K f(x-K)$$
$$= \Delta^K f(x-K) - \Delta^K f(x-K-1)$$
$$= \Delta^K [f(x-K) - f(x-(K+1))]$$
$$= \Delta^K [\Delta f(x-(K+1))]$$
$$= \Delta^{K+1} f(x-(K+1))$$

Hence the result is true for $r = K+1$

$\Delta^r f(x-r) = \nabla^r f(x)$

for all natural numbers r.

$\therefore \Delta^r f(x) = \Delta^r f(x-r)$.

Hence proved.

Prob : 16

Estimate the missing terms in the following table.

x	0	1	2	3	4
$u(x)$	1	3	9	-	81

Explain why the resulting value differ from 3^3.

Soln:

Let the missing term in $u(x)$ be 'a'.

since four values are known.

$$\Delta^4 u_0 = 0$$

Hence $(E-1)^4 u_0 = 0$ $\quad [\Delta = E-1]$

$$\therefore (E^4 - 4E^3 + 6E^2 - 4E + 1) u_0 = 0$$

$$E^4 u_0 - 4E^3 u_0 + 6E^2 u_0 - 4E u_0 + u_0 = 0$$

$$u_4 - 4u_3 + 6u_2 - 4u_1 + u_0 = 0$$

$$81 - 4a + 6(9) - 4(3) + 1 = 0$$

$$81 - 4a + 54 - 12 + 1 = 0$$

$$124 - 4a = 0$$

$$124 = 4a$$

$$a = 31$$

$\therefore u(x)$ the

a polynomia

polynomial

Prom : 17

Give

city for

given in

year 19

population
(in lakhs)

soln:

$u_4 =$

(E

=>

=>

=>

=>

=>

=>

∴ $u(3)$ The basic assumption is that $u(x)$ is a polynomial of degree 3, But 3^x is not a polynomial but an exponential function;

$$u(3) = 3^3 = 27.$$

Pbm : 17

Give an estimate of the population in a city for the year 1971 from the following data given in the table.

Year	1941	1951	1961	1971	1981	1991
Population (in lakhs)	363	391	421	?	467	501

soln:

Let a population in 1971 be 'a'.

Let $u_0 = 363$, $u_1 = 391$, $u_2 = 421$, $u_3 = a$, $u_4 = 467$, $u_5 = 501$

Five values are given $\Delta^5 u_0 = 0$

$$(E-1)^5 u_0 = (E^5 - 5C_1 E^4(1) + 5C_2 E^3(1)^2 - 5C_3 E^2 (1)^3 + 5C_4 E(1)^4 - 5C_5 E^0 (1)^5) u_0 = 0$$

$$\Rightarrow (E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1) u_0 = 0$$

$$\Rightarrow E^5 u_0 - 5E^4 u_0 + 10E^3 u_0 - 10E^2 u_0 + 5E u_0 - u_0 = 0$$

$$\Rightarrow u_5 - 5u_4 + 10u_3 - 10u_2 + 5u_1 - u_0 = 0$$

$$\Rightarrow 501 - 5(467) + 10a - 10(421) + 5(391) - 363 = 0$$

$$\Rightarrow 501 - 2335 + 10a - 4210 + 1955 - 363 = 0$$

$$\Rightarrow 10a - 4452 = 0$$

$$10a = 4452$$

$$a = 445.2 \text{ lakhs}$$

Pbm: 18.

Find the missing figures for the following table.

x	0	5	10	15	20	25
$u(x)$	7	11	?	18	?	32

soln:

Here two values are missing.

Let the missing values are 'a' and 'b'.

Let $u_0 = 7$, $u_1 = 11$, $u_2 = a$, $u_3 = 18$, $u_4 = b$, $u_5 = 32$

four values are given.

We have $\Delta^n u_x = 0$ for all $n \geq 4$ and for all x.

In particular, $\Delta^4 u_0 = 0$ and $\Delta^4 u_1 = 0$

Taking $\Delta^4 u_0 = 0$

$$(E-1)^4 u_0 = 0.$$

$$[E^4 - 4C_1 E^3 (1) + 4C_2 E^2 (1)^2 - 4C_3 E (1)^3 + 4C_4 E^0 (1)^4] u_0 = 0$$

$$(E^4 - 4E^3 + 6E^2 - 4E + 1) u_0 = 0$$

$$E^4 u_0 - 4E^3 u_0 + 6E^2 u_0 - 4E u_0 + u_0 = 0$$

$$u_4 - 4u_3 + 6u_2 - 4u_1 + u_0 = 0$$

$$b - 4(18) + 6a - 4(11) + 7 = 0$$

$$b - 72 + 6a - 44 + 7 = 0$$

$$6a + b - 109 \rightarrow ①$$

$$\Delta^4 u_1 = 0$$

$$(E^4 - 4C_1 E^3 + 4C_2 E^2 1^2 - 4C_3 E (1)^3 + 4C_4 E^0 (1)^4) u_1 = 0$$

$E^4 u_1 - 4E^3 u_1 + 6E^2 u_1 - 4E u_1 + u_1 = 0$

$u_5 - 4u_4 + 6u_3 - 4u_2 + u_1 = 0$

$32 - 4b + 6(18) - 4a + 11 = 0$

$-4a - 4b + 32 + 108 + 11 = 0$

$-4a - 4b + 151 = 0$

$4a + 4b - 151 = 0 \rightarrow (2)$

From (1) & (2),

(1) $\times 4 \Rightarrow 24a + 4b = 436$

(2) $\Rightarrow 4a + 4b = 151$

$\quad (-) \quad (-) \quad (-)$

$20a = 285$

$a = \frac{285}{20}$

$= \frac{57}{4}$

$a = 14.25$

sub in (2),

we get

$4a + 4b = 151$

$4(14.25) + 4b = 151$

$57 + 4b = 151$

$4b = 151 - 57$

$4b = 94$

$b = 23.5$

$\therefore$ The missing values a and b are 14.25 and 23.5 respectively.

Pbm: 19.

Given that $u_0+u_8 = 80$, $u_1+u_7 = 10$, $u_2+u_6 = 5$, $u_3+u_5 = 10$ find u_4.

soln:

Four values are given. we have

$\Delta^n u_x = 0$ for all $n \geq 4$.

$\Delta^8 u_x = 0$

$$[E^8 - 8C_1 E^7(1) + 8C_2 E^6(1)^2 - 8C_3 E^5(1)^3 + 8C_4 E^4(1)^4 - 8C_5 E^3(1)^5 + 8C_6 E^2(1)^6 - 8C_7 E(1)^7 + 8C_8 E^0(1)^8] u_0 = 0$$

$$(E^8 - 8E^7 + 28E^6 - 56E^5 + 70E^4 - 56E^3 + 28E^2 - 8E + 1) u_0 = 0$$

$$E^8 u_0 - 8E^7 u_0 + 28E^6 u_0 - 56E^5 u_0 + 70E^4 u_0 - 56E^3 u_0 + 28E^2 u_0 - 8E u_0 + u_0 = 0$$

$$u_8 - 8u_7 + 28u_6 - 56u_5 + 70u_4 - 56u_3 + 28u_2 - 8u_1 + u_0 = 0$$

$$u_0 + u_8 - 8(u_1+u_7) + 28(u_2+u_6) - 56(u_3+u_5) + 70u_4 = 0$$

$$80 - 8(10) + 28(5) - 56(10) + 70u_4 = 0$$

$$80 - 80 + 140 - 560 + 70u_4 = 0$$

$$-420 + 70u_4 = 0$$

$$70u_4 = 420$$

$$u_4 = 6$$

Pbm: 20

Given

$u_5 + u_6 = 113$

find u_{10}.

soln:

Pbm : 20

Given that $u_1 + u_2 + u_3 = 25$, $u_4 = 29$, $u_5 + u_6 = 113$ find polynomial $u(x)$ and hence find u_{10}.

Soln:

Since three values are given

$u(x)$ is a polynomial of degree 2.

Let $u(x) = ax^2 + bx + c$.

$u(1) = a(1)^2 + b(1) + c$

$u_1 = u(1) = a + b + c$

$u(2) = a(2)^2 + b(2) + c$

$u_2 = u(2) = 4a + 2b + c$

$u(3) = a(3)^2 + b(3) + c$

$u_3 = u(3) = 9a + 3b + c$

$u(4) = u_4 = a(4)^2 + b(4) + c$

$u_4 = 16a + 4b + c$

Given : $u_1 + u_2 + u_3 = 25$

$a + b + c + 4a + 2b + c + 9a + 3b + c = 25$

$14a + 6b + 3c = 25 \rightarrow (1)$

$u(5) = u_5 = a(5)^2 + b(5) + c$

$u_5 = 25a + 5b + c$

$u(6) = u_6 = a(6)^2 + b(6) + c$

$u_6 = 36a + 6b + c$.

Given : $u_4 = 29$

$16a + 4b + c = 29 \rightarrow (2)$

Given: $u_5 + u_6 = 113$

$25a + 5b + c + 36a + 6b + c = 113$

$$61a + 11b + 2c = 113 \rightarrow (3)$$

solve (1) & (2)

$$(1) \Rightarrow 14a + 6b + 3c = 25$$

$$(2) \times 3 \Rightarrow \underset{(-)}{48a} \underset{(-)}{+12b} \underset{(-)}{+3c} \underset{(-)}{=} 87$$

$$-34a - 6b = -62$$

$$34a + 6b = 62 \quad \rightarrow (4)$$

solve (2) & (3)

$$(2) \times 2 \Rightarrow 32a + 18b + 2c = 58$$

$$(3) \Rightarrow \underset{(-)}{61a} \underset{(-)}{+11b} \underset{(-)}{+2c} = \underset{(-)}{113}$$

$$-29a - 3b = -55$$

$$29a + 3b = 55 \rightarrow (5)$$

Solve (4) & (5).

$$(4) \Rightarrow 34a + 6b = 62$$

$$(5) \times 2 \Rightarrow \underset{(-)}{58a} \underset{(-)}{+6b} \underset{(-)}{=} 110$$

$$-24a = -48$$

$$a = 2$$

sub $a = 2$ in (5)

$$(5) \Rightarrow 29(2) + 3b = 55$$

$$3b = 55 - 58$$

$$3b = -3$$

$$b = -1$$

sub $a = 2, b = -1$ in (1)

$$(1) \Rightarrow 14(2) + 6(-1) + 3c = 25$$

$$28 - 6 + 3c = 25$$

$$3c = 25 - 22$$

$3c = 3$

$c = 1$

$u(x) = ax^2 + bx + c$

$u(x) = 2x^2 - x + 1$

$u(10) = u_{10} = 2(10)^2 - 10 + 1$

$= 2(100) - 10 + 1$

$= 200 - 10 + 1$

$u_{10} = 191$

Pbm : 21

If $u_1 = (12-x)(4+x)$; $u_2 = (5-x)(4-x)$; $u_3 = (x+18)(x+6)$ & $u_4 = 94$, obtain value of x assuming second order diff ~~of the function u(x) will be zero~~ all to be constant.

soln:

Since 2^{nd} order diff are assumed to be constant, the 3^{rd} order diff of the function,

$u(x)$ will all be zero.

$\therefore \Delta^3 u_1 = 0$. Hence $(E-1)^3 u_1 = 0$

$(E^3 - 3E^2 + 3E - 1)u_1 = 0$

$E^3 u_1 - 3E^2 u_1 + 3E u_1 - u_1 = 0$

$u_4 - 3u_3 + 3u_2 - u_1 = 0$

$94 - 3(x+18)(x+6) + 3(5-x)(4-x) - (12-x)(4+x) = 0$

$94 - 3(x^2 + 18x + 6x + 108) + 3(20 - 5x - 4x + x^2)$

$- (48 + 12x - 4x - x^2) = 0$

$94 - 3x^2 - 72x - (108)\cancel{3} + 3(20) - 27x + 3x^2$

$- 48 - 8x + x^2 = 0$

$x^2 - 107x + 94 - 324 + 60 - 48 = 0$

$x^2 - 107x - 218 = 0$

$x^2 - 109x + 2x - 218 = 0$

$x(x - 109) + 2(x - 109) = 0$

$(x - 109)(x + 2) = 0$

$x = 109, \quad x = -2$

Hence,

The value of x is 109 and -2.

Pbm: 22.

Evaluate $\frac{\Delta^2 x^3}{Ex^2}$ taking $h = 1$

<u>soln:</u>

$\Delta^2 x^3 = \Delta(\Delta x^3)$

$\Delta x^3 = (x + h)^3 - x^3$

Here $h = 1$

$= (x+1)^3 - x^3$

$= x^3 + 3x^2 + 3x + 1 - x^3$

$\Delta x^3 = 3x^2 + 3x + 1$

$\Delta^2 x^3 = \Delta(\Delta x^3)$

$= \Delta(3x^2 + 3x + 1)$

$= 3\Delta x^2 + 3\Delta x + \Delta(1)$

$= 3[(x+1)^2 - x^2] + 3[(x+1) - x] + 0$

$= 3[x^2 + 2x + 1 - x^2] + 3[x + 1 - x]$

$= 6x + 3 + 3$

$= 6x + 6$

$\Delta^2 x^3 = 6(x+1)$

Ex^2

$\frac{\Delta^2 x^3}{Ex^2}$

Pbm: 23

soln:

y

$E(x) = f(x+h)$

$$Ex^2 = (x+1)^2$$

$$\frac{\Delta^2 x^3}{Ex^2} = \frac{6(x+1)}{(x+1)^2}$$

$$= \frac{6}{x+1}$$

$$\therefore \quad \frac{\Delta^2 x^3}{Ex^2} = \frac{6}{x+1}.$$

Pbm: 23

Prove that $\left(\frac{\Delta^2}{E}\right)x^3 = 6x$.

Soln:

$$\left(\frac{\Delta^2}{E}\right)x^3 = \Delta^2 E^{-1}(x^3)$$

$$= \Delta^2 (x-1)^3$$

$$\left(\frac{\Delta^2}{E}\right)x^3 = \Delta^2 (x^3 - 3x^2 + 3x - 1) = \Delta^2 y$$

Let $y = x^3 - 3x^2 + 3x - 1$. Now express y as factorial polynomial.

$$\begin{array}{c|cccc}
 & 1 & -3 & 3 & -1 \\
0 & 0 & 0 & 0 & 0 \\
\hline
 & 1 & -3 & 3 & \boxed{-1} \\
1 & 0 & 1 & -2 & \\
\hline
 & 1 & -2 & \boxed{1} & \\
2 & 0 & 2 & & \\
\hline
 & 1 & \boxed{0} & &
\end{array}$$

$$\therefore y = x^{(3)} + x^{(1)} - 1$$

$$\Delta y = \Delta\left[x^{(3)} + x^{(1)} - 1\right]$$

$$\Delta y = 3x^{(2)} + 1$$

$\therefore \Delta^2 y = 3 \cdot 2x^{(1)} + 0$

$= 6x^{(1)}$

$\Delta^2 y = 6x = \left(\frac{\Delta^2}{E}\right) x^3$

Hence, $\left(\frac{\Delta^2}{E}\right) x^3 = 6x$

Pbm: 24.

Prove that

i) $\nabla^2 y_8 = y_8 - 2y_7 + y_6$

ii) $\delta^2 y_5 = y_6 - 2y_5 + y_4$

soln:

i) $\nabla^2 y_8 = \nabla(\nabla y_8)$

$= \nabla(y_8 - y_7)$

$= \nabla y_8 - \nabla y_7$

$= (y_8 - y_7) - (y_7 - y_6)$

$= y_8 - y_7 - y_7 + y_6$

$\nabla^2 y_8 = y_8 - 2y_7 + y_6$

ii) $\delta^2 y_5 = \delta(\delta y_5)$

$= \delta(y_{5+1/2} - y_{5-1/2})$

$= \delta(y_{11/2} - y_{9/2})$

$= \delta y_{11/2} - \delta y_{9/2}$

$= (y_{11/2+1/2} - y_{11/2-1/2}) - (y_{9/2+1/2} - y_{9/2-1/2})$

$= (y_{12/2} - y_{10/2}) - (y_{10/2} - y_{8/2})$

$= y_6 - y_5 - y_5 + y_4$

$\delta^2 y_5 = y_6 - 2y_5 + y_4$

pbm : 25

Explain the diff between $\left(\frac{\Delta^2}{E}\right) f(x)$ & $\frac{\Delta^2 f(x)}{E f(x)}$ and find the value of these when $f(x) = x^2$.

Soln: Take h as the interval of difference

$$\left(\frac{\Delta^2}{E}\right) f(x) = \frac{\Delta^2}{E} x^2$$

$$= \Delta^2 E^{-1} x^2$$

$$= \Delta^2 (x-h)^2$$

$$= \Delta^2 (x^2 + h^2 - 2xh)$$

$$= \Delta (\Delta x^2 + \Delta h^2 - 2\Delta xh)$$

$$= \Delta ((x+h)^2 - x^2 - 2h[(x+h) - x]$$

$$= \Delta (x^2 + h^2 + 2hx - x^2 - 2hx - 2h^2 + 2hx)$$

$$= \Delta (2hx - h^2)$$

$$= 2h\,\Delta(x) - \Delta h^2$$

$$= 2h[(x+h) - x]$$

$$= 2hx + 2h^2 - 2hx$$

$$\left(\frac{\Delta^2}{E}\right) f(x) = 2h^2 \rightarrow (1)$$

$$\frac{\Delta^2 f(x)}{E f(x)} = \frac{\Delta^2 x^2}{E x^2}$$

$$\Delta^2 x^2 = \Delta (\Delta x^2)$$

$$= \Delta[(x+h)^2 - x^2]$$

$$= \Delta [x^2 + h^2 + 2xh - x^2]$$

$= \Delta(h^2 + 2xh)$

$= \Delta h^2 + 2h\,\Delta x$

$= 2h\,[(x+h) - x]$

$= 2h\,[x+h-x]$

$= 2h\,(h)$

$\Delta^2 x^2 = 2h^2$

$E x^2 = (x+h)^2$

$$\frac{\Delta^2 f(x)}{E f(x)} = \frac{2h^2}{(x+h)^2} \rightarrow ②$$

From ① & ②

$$\left(\frac{\Delta^2}{E}\right) f(x) \neq \frac{\Delta^2 f(x)}{E f(x)}.$$

Unit 3

Unit – III

Numerical Differentiation and Integration – Introduction, Numerical Differentiation – Errors in Numerical Differentiation – Cubic Spline method – maximum and minimum values of a tabulated function, Numerical Integration – Trapezoidal Rule and Simpson's 1/3 and 3/8 rules.

Unit - iii

INTERPOLATION

Newton's Interpolation formulae.

i) Newton's forward interpolation formula for equal intervals:

$$y(x) = y_p = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \cdots + \frac{P(P-1)\cdots(P-\overline{n-1})}{n!}\Delta^n y_0.$$

(where $P-\overline{n-1} = P-(n-1)$)

where, $x = x_0 + Ph$ or $P = \frac{x-x_0}{h}$.

ii) Newton's backward interpolation formula:

$$y(x) = y_n + P\nabla y_n + \frac{P(P+1)}{2!}\nabla^2 y_n + \frac{P(P+1)(P+2)}{3!}\nabla^3 y_n + \cdots + \frac{P(P+1)\cdots(P+\overline{n-1})}{n!}\nabla^n y_n$$

where $x = x_n + Ph$.

Pbm : 1

If $y(75) = 246$, $y(80) = 202$, $y(85) = 118$, $y(90) = 40$ find $y(79)$.

Soln:

Here $x_0 = 75$, $h = 5$ & to find value of y at $x = 79$.

Newton's forward interpolation formula is

$$y_p = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \cdots + \frac{P(P-1)\cdots(P-(n-1))}{n!}\Delta^n y_0 \quad \rightarrow (1)$$

where $p = \frac{x - x_0}{h}$ $(x = 79, x_0 = 75)$

$$p = \frac{79 - 75}{5}$$

$$p = \frac{4}{5} = 0.8.$$

form the forward difference table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
75	246			
		-44		
80	202		-40	46
		-84		
85	118		6	
		-78		
90	40			

Using values of Δy_0, $\Delta^2 y_0$ & $\Delta^3 y_0$ in ①,

$$y_{0.8} = 246 + 0.8(-44) + \frac{0.8(0.8-1)}{2!}(-40) + \frac{(0.8)(0.8-1)(0.8-2)}{3!}(46)$$

$$= 246 - 35.2 + \frac{(0.8)(-0.2)}{1\times 2}(-40) + \frac{(0.8)(-0.2)(-1.2)(46)}{1\times 2\times 3}$$

$$= 246 - 35.2 + 3.2 + 1.472$$

$$y_{0.8} = 215.472$$

$$f(0.8) = 215.472.$$

Pbm: 2.

Find a cube polynomial, which takes the following data.

x	0	1	2	3
$f(x)$	1	2	1	10

soln

Form the difference table :

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
0	1			
		1		
1	2		-2	
		-1		12
2	1		10	
		9		
3	10			

By Newton's - Gregory formula,

$$f(x) = f(x_0) + (x-x_0)\frac{\Delta f(x_0)}{1!\,h} + (x-x_0)(x-x_0-h)\frac{\Delta^2 f(x_0)}{2!\,h^2} + \cdots$$

Here $x_0 = 0$ & $h = 1$ (Here $p = \frac{x-x_0}{h}$)

$y(x) = y_0 + p\frac{\Delta y_0}{1!} + \frac{p(p-1)}{2!}\Delta^2 y_0 +$

$$f(x) = f(0) + (x-0)\frac{\Delta f(0)}{1!\,(1)} + \frac{(x-0)(x-0-1)}{2!\,(1)^2}\Delta^2 f(0) + \cdots$$

$$= 1 + (x-0)\frac{1}{1!} + (x-0)(x-1)\frac{(-2)}{2!} + (x-0)(x-1)(x-2)\frac{12}{3!}$$

$$= 1 + x + x(x-1)(-1) + x(x-1)(x-2)(2)$$

$$= 1 + x - x(x-1) + 2(x)(x-1)(x-2)$$

$$= 1 + x - x^2 + x + 2x(x^2 - 2x - x + 2)$$

$$= 1 + x - x^2 + x + 2x^3 - 4x^2 - 2x^2 + 4x$$

$$= 1 + 6x - 7x^2 + 2x^3$$

$$f(x) = 2x^3 - 7x^2 + 6x + 1$$

Pbm 3

A fn $y = f(x)$ is given by following table. find $f(0.2)$ by a suitable formula

x	0	1	2	3	4	5	6
$y=f(x)$	176	185	194	203	212	220	229

Soln:

Since the value $x = 0.2$ is near the beginning of the table.

So, Newton's forward interpolation formula.

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots + \frac{p(p-1)\cdots(p-(n-1))}{n!}\Delta^n y_0 \quad \to (1)$$

where $p = \dfrac{x - x_0}{h}$

Here $x_0 = 0$ & $h = 1$ and to find the value of $f(x)$ at $x = 0.2$

$$\therefore p = \frac{0.2 - 0}{1} = 0.2.$$

form the forward difference table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$	$\Delta^6 f(x)$
0	176						
		9					
1	185		0				
		9		0			
2	194		0		0		
		9		0		-1	
3	203		0		-1		5
		9		-1		4	
4	212		-1		3		
		8		2			
5	220		1				
		9					
6	229						

① becomes

$$y_{0.2} = 176 + 0.2(9) + \frac{(0.2)(0.2-1)(0.2-2)(0.2-3)(0.2-4)}{5!}(-1) + \frac{(0.2)(0.2-1)(0.2-2)(0.2-3)(0.2-4)(0.2-5)}{6!}(5)$$

$$= 176 + 1.8 + \frac{(0.2)(-0.8)(-1.8)(-2.8)(-3.8)}{120}(-1) + \frac{(0.2)(-0.8)(-1.8)(-2.8)(-3.8)(-4.8)}{720}(5)$$

$$= 176 + 1.8 - \frac{3.0643}{120} \cancel{-\frac{75.0931}{720}} \; \frac{73.5437}{720}$$

$$= 176 + 1.8 - 0.0255 - 0.1021$$

$$= 177.6724$$

$$f(0.2) = 177.67$$

Pbm: 4

construct Newton's forward interpolation polynomial for the following data.

x	4	6	8	10
y	1	3	8	16

Use it to find the value of y for $x = 5$.

Soln:

Here $x_0 = 4$ and $h = 2$.

Newton's forward interpolation formula is

$$y(x) = y_0 + \frac{x - x_0}{1!\,h}\Delta y_0 + (x - x_0)(x - x_0 - h)\frac{\Delta^2 y_0}{2!\,h^2} + \dots \rightarrow ①$$

form the difference table:-

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
4	1			
		2		
			3	
6	3			0
		5		
8	8		3	
		8		
10	16			

$$\therefore y(x) = 1 + \frac{(x-4)\,2}{1! \times 2} + \frac{(x-4)(x-6)}{2!} \times \frac{3}{2^2}$$

$$+ \frac{(x-4)(x-6)(x-8)}{3!} \times \frac{0}{2^3}$$

$$= 1 + \frac{x-4}{1!} + \frac{3}{4} \cdot \frac{(x-4)(x-6)}{2} + 0$$

$$= 1 + (x-4) + \frac{3(x-4)(x-6)}{8}$$

$$= 1+(x-4)+ \frac{3(x^2-4x-6x+24)}{8}$$

$$= 1+(x-4)+ \frac{3x^2-12x-18x+72}{8}$$

$$= 1+(x-4)+ \frac{3x^2-30x+72}{8}$$

$y(x) = 1+(x-4) + \frac{3}{8}(x^2-10x+24)$ is the required interpolating polynomial.

when $x=5$,

$$y_5 = 1 + (5-4) + \frac{3}{8}(5^2 - 10(5) + 24)$$

$$= 1+1+\frac{3}{8}(25-50+24)$$

$= 2 + \frac{3}{8}(-1)$

$= 2 - 3/8$

$= \frac{16-3}{8}$

$= \frac{13}{8}$

$y_5 = 1.625$

Pbm: 5

Find the value of y from the following data at $x = 2.65$.

x	-1	0	1	2	3
y	-21	6	15	12	3

Soln:

Since, value of $x = 2.65$ is between 2 and 3 (at the end of the table).

Use Newton's backward interpolation formula:

$$y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \dots$$

where $p = \frac{x - x_n}{h}$

Here $x = 2.65$, $x_n = 3$, $h = 1$

$$p = \frac{2.65 - 3}{1} = -0.35$$

To find: ∇y_n, $\nabla^2 y_n$ etc... we form backward difference table as

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
-1	-21				
		27			
0	6		-18		
				6	
1	15	9	-12		0
2	12	-3		6	
			-6		
3	3	-9			

$$y_{-0.35} = 3 + (-0.35)(-9) + \frac{(-0.35)(-0.35+1)}{2!}(-6)$$

$$+ \frac{(-0.35)(-0.35+1)(-0.35+2)}{3!}(6) + 0$$

$$= 3 + 3.15 + \frac{(0.35)(0.65)(6)}{2} - \frac{(0.35)(0.65)(1.65)}{6}$$

$$= 6.15 + 0.6825 - 0.3754$$

$$= 6.4571$$

$$y_{-0.35} = 6.4571$$

Pbm : 6

Soln:

i) Since $x=48$ is near the beginning of the table.

Use Newton's forward interpolation formula as

$$\theta_p = \theta_0 + p\Delta\theta_0 + \frac{p(p-1)}{2!}\Delta^2\theta_0 + \dots$$

where $p = \frac{x-x_0}{h} = \frac{48-40}{10} = \frac{8}{10}$

$p = 0.8$ (Here $x_0 = 40$, $h=10$)

To find $\Delta\theta_0$, $\Delta^2\theta_0$ etc, we form forward difference table.

x	θ	$\Delta\theta$	$\Delta^2\theta$	$\Delta^3\theta$	$\Delta^4\theta$	$\Delta^5\theta$
40	184					
		20				
50	204		2			
		22		0		
60	226		2		0	
		24		0		0
70	250		2		0	
		26		0		
80	276		2			
		28				
90	304					

forward

backward

$$\theta_{0.8} = 184 + 0.8(20) + \frac{(0.8)(0.8-1)}{2!}\times 2$$

$$= 184 + 16 + \frac{(0.8)(-0.2)}{2}\times 2$$

~~= 184 + 16 - 0.48~~

$$= 184 + 16 - 0.16$$

$$\theta_{0.8} = 199.84$$

$$\theta_{0.8} \approx 200$$

ii) Since $x = 84$ is nearer to the end of the table.

we use Newton's backward interpolation formula

$$\theta_p = \theta_n + p\nabla\theta_n + \frac{p(p+1)}{2!}\nabla^2\theta_n + \dots$$

where $p = \frac{x - x_n}{h}$

$= \frac{84-90}{10}$ (Here $x_n = 90$, $h = 10$)

$p = -\frac{6}{10} = -0.6$

$$\theta_{-0.6} = 304 + (-0.6)\times 28 + \frac{(-0.6)(-0.6+1)}{2!}(2)$$

$= 304 - 16.8 + \frac{(-0.6)(0.4)}{2}\times 2$

$= 304 - 16.8 - 0.24$

$\theta_{-0.6} = 286.96$

$\theta_{0.6} \approx 287$

[Value of $\theta_n = 304$, $\nabla\theta_n = 28$, $\nabla^2\theta_n = 2$, $\nabla^3\theta_n = \nabla^4\theta_n = \nabla^5\theta_n = 0$ from forward diff table itself by sloping backward]

Pbm: 7

Soln:

The Less than cumulative frequency table of the given data is

Weight less than x	40	60	80	100	120
No. of students	250	370	470	540	590

Form the diff. table:-

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	250				
		120			
60	370		-20		
		100		-10	
80	470		-30		20
		70		10	
100	540		-20		
		50			
120	590				

No. of students whose weight is between 60 and 70 is get from the expression $y_{70} - y_{60}$. We have $y_{60} = 370$ (from the table)

To find y_{70} by Newton's Interpolation formula.

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \cdots + \frac{p(p-1)\cdots(p-(n-1))}{n!}$$

where $p = \frac{x - x_0}{h}$

$x = 70,\ x_0 = 40,\ h = 20$

$$p = \frac{70-40}{20} = \frac{30}{20} = 1.5$$

$$y_{70} = y_{1.5} = 250 + 1.5(120) + \frac{(1.5\times 0.5)}{2!}(-20)$$

$$+ \frac{(1.5)(0.5)(-0.5)}{3!}(40) + \frac{(1.5)(0.5)(-0.5)(-1.5)}{4!}(20)$$

$$= 250 + 180 - 7.5 + 0.625 + 0.4688$$

$$y_{70} = 423.5938$$

$\therefore$ $y_{70} = 424$ (approximately)

$\therefore$ No. of students whose weight is between 60 and 70 is

$$= y_{70} - y_{60}$$

$$= 424 - 370$$

$$= 54$$

Central Difference Interpolation formulae.

i) Gauss forward Interpolation formula

ii) Gauss backward Interpolation formula

iii) Sterling's formula.

i) Gauss forward interpolation formula

$$y_p = y_0 + \binom{p}{1}\Delta y_0 + \binom{p}{2}\Delta^2 y_{-1} + \binom{p+1}{3}\Delta^3 y_{-1} + \binom{p+1}{4}\Delta$$

$$+ \binom{p+2}{5}\Delta^5 y_{-2} + \binom{p+2}{6}\Delta^6 y_{-3} + \cdots\cdots$$

Gauss - Backward Interpolation formula

$$y_p = y_0 + \binom{p}{1}\Delta y_{-1} + \binom{p+1}{2}\Delta^2 y_{-1} + \binom{p+1}{3}\Delta^3 y_{-2} + \binom{p+2}{4}\Delta^4 y_{-2} + \cdots$$

Stirling's formula

$$y_p = y_0 + p\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right] + \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \cdots$$

Pbm: 1

Apply Gauss forward interpolation formula to find y(25) for the following data.

x	20	24	28	32
y	2854	3162	3544	3992

Soln:

Here $h=4$. To find y(25) take the origin $x_0 = 24$

At $x = 25$, $p = \frac{x - x_0}{h} = \frac{25-24}{4} = \frac{1}{4}$

$p = 0.25$

Apply Gauss forward formula,

$$y_p = y_0 + \binom{p}{1}\Delta y_0 + \binom{p}{2}\Delta^2 y_{-1} + \binom{p+1}{3}\Delta^3 y_{-1} + \cdots$$

Central difference table :

x	$p = \frac{x-24}{4}$	y	Δy	$\Delta^2 y$	$\Delta^3 y$
20	-1	2854			
			308		
24	0	3162		74	-8
			382		
				66	
28	1	3544			
			448		
32	2	3992			

$$y_{0.25} = 3162 + 0.25(382) + \frac{(0.25)(0.25-1)}{2!} \times 74 + \frac{(0.25+1)(0.25)(0.25-1)}{3!} \times (-8)$$

$$= 3162 + 95.5 + (0.25)(-0.75)(37) + \frac{(1.25)(0.25)(-0.75)}{6} \times (-8)$$

$$= 3162 + 95.5 - 6.9375 + 0.3125$$

$$= 3250.875$$

$y_{0.25} = 3251$

$y(25) = 3251$

Pbm 2

Apply Gauss backward interpolation formula to find $y(25)$ in (Pbm : 1)

Soln:

Here $h = 4$. To find $y(25)$ take origin $x_0 = 28$

At $x = 25$, $p = \frac{25-28}{4} = \frac{-3}{4} = -0.75$

Gauss backward interpolation formula is

$$y_p = y_0 + \binom{p}{1}\Delta y_{-1} + \binom{p+1}{2}\Delta^2 y_{-1} + \binom{p+1}{3}\Delta^3 y_{-2} +$$

central diff. table.

x	$p = \frac{x-28}{4}$	y	Δy	$\Delta^2 y$	$\Delta^3 y$
20	-2	2854			
			308		
24	-1	3162		74	
			382		-8
28	0	3544		66	
			448		
32	1	3992			

$$y_{0.75} = 3544 + (-0.75)(382) + \frac{(-0.75+1)(-0.75)}{2} \times 66 + \frac{(-0.75+1)(-0.75)(-0.75-1)}{6}(-8)$$

$$= 3544 - 286.5 + \frac{(0.25)(-0.75)}{2} \times 66 + \frac{(0.25)(-0.75)(-1.75)}{6}(-8)$$

$$= 3544 - 286.5 - 6.1875 - 0.4375$$

$$= \cancel{3250.88} = 3250.88$$

$y(25) = 3251$.

Pbm: 3.

soln: Here $h = 10$

i) To find y_{15}, we take origin at $x_0 = 20$ & apply Gauss backward interpolation formula.

Here $p = \frac{x - x_0}{h} = \frac{15-20}{10} = \frac{-5}{10} = -0.5$

we have to find $y_{-0.5}$

x°	$p=\frac{x-x_0}{h}$	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	-2	0.51					
			0.04				
				-0.02			
10	-1	0.55			0.02		
			0.02			-0.01	
20	0	0.57		0			0.01
			0.02		0.01		
30	1	0.59		0.01		0	
			0.03		0.01		
40	2	0.62		0.02			
			0.05				
50	3	0.67					

By Gauss backward formula,

$$y_p = y_0 + \binom{p}{1}\Delta y_{-1} + \binom{p+1}{2}\Delta^2 y_{-1} + \binom{p+1}{3}\Delta^3 y_{-2} + \binom{p+2}{4}\Delta^4 y_{-2} + \binom{p+2}{5}\Delta^5 y_{-3} + \cdots$$

At $p = -0.5$

$$y_{-0.5} = 0.57 + \frac{(-0.5)(0.02)}{1} + \frac{(-0.5+1)(-0.5)}{2}\times 0 + \frac{(-0.5+1)(-0.5)(-0.5-1)}{6}(0.02) + \frac{(-0.5+2)(-0.5+1)(-0.5)(-0.5-1)}{24}(-0.02) + \frac{(-0.5+2)(-0.5+1)(-0.5)(-0.5-1)(-0.5-2)}{120}\times 0.01$$

$$= 0.57 - 0.01 + \frac{(0.5)(-0.5)}{2}\times 0 + \frac{(0.5)(-0.5)(-1.5)}{6}(0.02) + \frac{(1.5)(0.5)(-0.5)(-1.5)}{24}(-0.02)$$

$$= 0.57 - 0.01 + 0 + 0.00125 - 0.0005$$

$$= 0.56075$$

$$\simeq 0.5610$$

ii) To find value of y at $x = 45$, Newtons backward interpolation formula, in the above table.

$$y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \cdots$$

$$p = \frac{x - x_n}{h} = \frac{45-50}{10} = -0.5$$

$$y_{-0.5} = 0.67 + (0.5)(0.05) + \frac{(-0.5)(-0.5+1)}{2!}(0.02)$$

$$+ \frac{(-0.5)(-0.5+1)(-0.5+2)}{3!}(0.01)$$

$$+ \frac{(-0.5)(-0.5+1)(-0.5+2)(-0.5+3)(-0.5+4)}{5!}(0.0)$$

$$= 0.67 + 0.025 \underset{\cancel{0.0005}}{-0.0025} - 0.0001 - 0.0013$$

$$= 0.6416.$$

Pbm: 7.

~~Pbm: 4~~

Soln

Here $h = 4$, To find $y(25)$ take origin at $x_0 = 24$.

At $x = 25$, $P = \frac{x - x_0}{h}$

$$P = \frac{25-24}{4} = \frac{1}{4} = 0.25$$

Stirling formula is

$$y_p = y_0 + P\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{P(P^2-1)}{3!}$$

$$\left[\frac{\Delta^3 y_{-1} + \Delta y_{-2}}{2}\right] + \cdots$$

central difference table

x	$P = \frac{x-24}{4}$	y	Δy	$\Delta^2 y$	$\Delta^3 y$
20	-1	2854			
			308		
24	0	3162		74	
			382		-8
28	1	3544		66	
			448		
32	2	3992			

$$y_{0.25} = 3162 + 0.25\left(\frac{382+308}{2}\right) + \frac{(0.25)^2}{2!} \times 74$$

$$= 3162 + 0.25\left(\frac{690}{2}\right) + \frac{0.0625}{2} \times 74$$

$$= 3162 + 0.25(345) + 2.3125$$

$$= 3162 + 86.25 + 2.3125$$

$$= 3250.5625$$

$y(25) \approx 3251$

Pbm: 5

Using stirling's formula compute y_{35} given that $y_{10} = 600$, $y_{20} = 512$, $y_{30} = 439$, $y_{40} = 346$, $y_{50} = 243$.

Soln: Here $h = 10$. To find y_{35} take $x_0 = 30$

At $x = 35$, $P = \frac{x - x_0}{h}$

$$P = \frac{35-30}{10} = 0.5$$

central difference table

x	$P = \frac{x-30}{10}$	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
10	-2	600				
			-88			
20	-1	512		15		
			-73		-35	
30	0	439		-20		45
			-93		10	
40	1	346		-10		
			-103			
50	2	243				

Stirling's formula is

$$y_p = y_0 + P\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{P^2}{2!}\Delta^2 y_{-1} + \frac{P(P^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta y_{-2}}{2}\right) + \frac{P^2(P^2-1^2)}{4!}\Delta^4 y_{-2}$$

$$y_{0.5} = 439 + 0.5\left(\frac{(-93)+(-73)}{2}\right) + \frac{(0.5)^2}{2!}(-20) + \frac{0.5(0.5^2-1^2)}{3!}\left(\frac{10+(-35)}{2}\right) + \frac{0.5^2}{4!}(0.5^2-1^2) \times 45$$

$$= 439 - 41.5 - 2.5 + 0.7813 - 0.3516$$

$$= 395.4297$$

$\therefore y(35) = 395$

Lagrange's Interpolation formula

$$y = f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} y_1$$

$$+ \cdots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} y_n$$

Pbm: 1

x	3	7	9	10
y	168	120	72	63

soln

Lagrange's formula for four sets of data is

$$y = \left[\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\right] y_0 + \left[\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}\right] y_1$$

$$+ \left[\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}\right] y_2 + \left[\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}\right] y_3$$

$$= \frac{(6-7)(6-9)(6-10)}{(3-7)(3-9)(3-10)} \times 168 + \left[\frac{(6-3)(6-9)(6-10)}{(7-3)(7-9)(7-10)}\right] \times 120 + \left[\frac{(6-3)(6-7)(6-10)}{(9-3)(9-7)(9-10)}\right] \times 72$$

$$+ \left[\frac{(6-3)(6-7)(6-9)}{(10-3)(10-7)(10-9)}\right] \times 63$$

$$= \frac{-12}{-168} \times 168 + \frac{36}{24} \times 120 + \frac{12}{-12} \times 72 + \frac{9}{21} \times 63$$

~~= 12 + 180 + 180 +~~

$$= 12 + 180 - 72 + 27$$

$y = 147$

Pbm: 2

x	0	1	3	4
y	-12	0	6	12

find the value of y when $x = 2$.

soln:

Lagrange's formula for four set of data is

$$y = \left[\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\right] y_0 + \left[\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}\right] y_1 + \left[\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}\right] y_2 + \left[\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}\right] y_3$$

Here $x_0 = 0,\ x_1 = 1,\ x_2 = 3,\ x_3 = 4$

$y_0 = -12,\ y_1 = 0,\ y_2 = 6,\ y_4 = 12$

$$y = \left[\frac{(x-1)(x-3)(x-4)}{(-1)(-3)(-4)}\right](-12) + \left[\frac{(x-0)(x-3)(x-4)}{(1)(-2)(-3)}\right] \times 0 + \left[\frac{(x-0)(x-1)(x-4)}{(3)(+2)(-1)}\right](6) + \left[\frac{(x-0)(x-1)(x-3)}{4\times 3\times 1}\right] \times 12$$

$$= \frac{(x-1)(x-3)(x-4)}{-12}(-12) + 0 + \frac{x(x-1)(x-4)}{-6}(6) + \frac{x(x-1)(x-3)}{12} \times 12$$

$$= (x-1)\left[(x-3)(x-4) - x(x-4) + x(x-3)\right]$$

$$= (x-1)\left[x^2 - 3x - 4x + 12 - x^2 + 4x + x^3 - 3x\right]$$

$$= (x-1)\left[x^2 - 6x + 12\right]$$

$= x^3 - 6x^2 + 12x - x^2 + 6x - 12$

$y = x^3 - 7x^2 + 18x - 12.$

$y(x) = x^3 - 7x^2 + 18x - 12$

at $x = 2$, $y(2) = 2^3 - 7(2^2) + 18(2) - 12$

$= 8 - 7(4) + 36 - 12$

$= 8 - 28 + 24$

$= 8 - 4$

$y(2) = 4$

Pbm: 3 Find y form & find $y(3)$

x	0	1	2	5
y	2	3	12	147

Soln:

$x_0 = 0,\ x_1 = 1,\ x_2 = 2,\ x_3 = 5$

$y_0 = 2,\ y_1 = 3,\ y_2 = 12,\ y_3 = 147$

Applying Lagrange's formula,

$$y = \left[\frac{(x-1)(x-2)(x-5)}{(-1)(-2)(-5)}\right] \times 2 + \left[\frac{x(x-2)(x-5)}{(1)(-1)(-4)}\right] \times 3$$

$$+ \left[\frac{x(x-1)(x-5)}{(2)(1)(-3)}\right] \times 12 + \left[\frac{x(x-1)(x-2)}{(5)(4)(3)}\right] \times 147$$

$$= \left[\frac{-(x-1)(x^2-7x+10)}{5}\right] + \frac{3(x^3-7x^2+10x)}{4}$$

$$-2(x^3-6x^2+5x) + \frac{x^3-3x^2+2x}{60} \times 147$$

$$= \frac{-(x^3-8x^2+17x-10)}{5} + \frac{3(x^3-7x^2+10x)}{4} - 2(x^3-6x^2+5x)$$

$$+ \frac{x^3-3x^2+2x}{60} \times 147$$

$= \frac{1}{60}(60x^3 + 60x^2 - 60x + 120)$

$y = x^3 + x^2 - x + 2$

At $x = 3$, $y(3) = 3^3 + 3^2 - 3 + 2$

$= 27 + 9 - 3 + 2$

$y(3) = 35$

Pbm: 4

Value of $U(x)$ are known at a, b, c. S.T maximum or minimum of Lagrange's Interpolation polynomial is attained by.

$$x = \frac{\sum U_a (b^2 - c^2)}{2 \sum U_a (b - c)}$$

Soln: By Lagrange's formula for 3 sets of values,

$$U(x) = \frac{(x-b)(x-c)}{(a-b)(a-c)} \times U_a + \frac{(x-a)(x-c)}{(b-a)(b-c)} \times U_b + \frac{(x-a)(x-b)}{(c-a)(c-b)} \times U_c.$$

$$= \sum \left[\frac{x^2 - (b+c)x + bc}{(a-b)(a-c)} \right] U_a$$

$U(x)$ attains its maximum or minimum when $\frac{dU}{dx} = 0$.

Now, $\frac{du}{dx} = 0 \Rightarrow \sum \left[\frac{2x - (b+c)}{(a-b)(a-c)} \right] U_a = 0$

$$\Rightarrow \left[\frac{2x - (b+c)}{(a-b)(a-c)} \right] U_a + \left[\frac{2x - (c+a)}{(b-c)(b-a)} \right] U_b + \left[\frac{2x - (a+b)}{(c-a)(c-b)} \right] U_c = 0$$

multiply by $(a-b)(b-c)(a-c)$

$$\Rightarrow (b-c)[2x-(b+c)]U_a + (c-a)[2x-(c+a)]U_b + (a-b)[2x-(a+b)]U_c = 0$$

$$\Rightarrow 2x[U_a(b-c) + U_b(c-a) + U_c(a-b)] = (b^2-c^2)U_a + (c^2-a^2)U_b + (a^2-b^2)U_c$$

$$x = \frac{(b^2-c^2)U_a + (c^2-a^2)U_b + (a^2-b^2)U_c}{2[U_a(b-c)+U_b(c-a)+U_c(a-b)]}$$

$$\therefore x = \frac{\Sigma U_a(b^2-c^2)}{2\Sigma U_a(b-c)}$$

Divided Difference.

1st divided differences

$$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0}$$

2nd divided differences.

$$[x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}$$

3rd divided differences.

$$[x_0, x_1, x_2, x_3] = \frac{[x_1, x_2, x_3] - [x_0, x_1, x_2]}{x_3 - x_0}$$

Pbm: 1 If $f(x) = \frac{1}{x^2}$ find the 1st divided differences.

i) $[a, b]$ ii) $[a, b, c]$

Soln:

Here a, b, c are arguments for

$f(x) = \frac{1}{x^2}$

$\therefore f(a) = \frac{1}{a^2}$; $f(b) = \frac{1}{b^2}$; $f(c) = 1/c^2$

i) $[a, b] = \frac{f(b) - f(a)}{b-a} = \frac{1/b^2 - 1/a^2}{b-a}$

$= \frac{a^2 - b^2}{a^2 b^2 (b-a)}$

$= \frac{(a+b)(a-b)}{a^2 b^2 (b-a)}$

$= -\frac{a+b}{a^2 b^2}$

ii) $[a, b, c] = \frac{[b, c] - [a, b]}{c-a}$

$= \frac{\left\{-\frac{(b+c)}{b^2 c^2}\right\} - \left\{-\frac{a+b}{a^2 b^2}\right\}}{c-a}$

$= \frac{1}{c-a}\left(\frac{c^2(a+b) - a^2(b+c)}{a^2 b^2 c^2}\right)$

$= \frac{c^2 a + c^2 b - a^2 b - a^2 c}{a^2 b^2 c^2 (c-a)}$

$= \frac{1}{c-a}\left(\frac{ac(c-a) + b(c^2 - a^2)}{a^2 b^2 c^2}\right)$

$= \frac{ab+bc+ca}{c-a}$

Pbm: 2

Find the divided diff. table for the function $f(x) = x^2 + 2x + 2$ whose arguments are 1, 2, 4, 7, 10

Soln:

When $x = 1, 2, 4, 7, 10$, the values of given function are 5, 10, 26, 65, 122. respectively.

Divided difference table is as shown below.

x	$f(x)$	1st div. diff	2nd div. diff	3rd div diff	4th div diff.
1	5				
		$\frac{10-5}{2-1} = 5$			
2	10		$\frac{8-5}{4-1} = 1$		
		$\frac{26-10}{4-2} = 8$		0	0
4	26		$\frac{13-8}{7-2} = 1$		
		$\frac{65-26}{7-4} = 13$			
7	65			0	
			$\frac{19-13}{10-4} = 1$		
10	122	$\frac{122-65}{10-7} = 19$			

Newton's divided diff. formula

$$f(x) = f(x_0) + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + \dots + (x - x_0)\dots(x - x_n)[x_0, x_1, \dots, x_n]$$

Pbm: 1)

soln: Newton's divided difference formula is

$f(x) = f(x_0) + (x-x_0)[x_0,x_1] + (x-x_0)(x-x_1)[x_0,x_1,x_2] + \dots$ ×6

At $x=8$,

$f(8) = 48 + (8-4)[4,5] + (8-4)(8-5)[4,5,7]$
$+ (8-4)(8-5)(8-7)[4,5,7,10] + (8-4)(8-5)$
$(8-7)(8-10)[4,5,7,10,11] + (8-4)(8-5)(8-7)$
$(8-10)(8-11)[4,5,7,10,11,13]$.

x	$f(x)$	1st div diff	2nd div diff	3rd div diff	4th div diff	5th div diff
4	48					
		52				
5	100		15			
				1		
7	294	97	21		0	
10	900	202		1		0
			27			
11	1210	310			0	
			33	1		
13	2028	409				

sub the values of all divided diff from divided diff. table,

$f(8) = 48 + (4\times52) + (4\times3\times15) + (4\times3\times1\times1)$
$+ (4\times3\times1\times-2)(0) + 0$

$= 48 + 208 + 180 + 12$

$f(8) = 448$

Pbm: 2

Soln:

To find the eqn of cubic curve from the following data.

x	4	7	9	12
$f(x)$	-43	83	327	1053.

Apply Newton's divided difference formula

$$f(x) = f(x_0) + (x-x_0)[x_0, x_1] + (x-x_0)(x-x_1)[x_0, x_1, x_2] + (x-x_0)(x-x_1)(x-x_2)[x_0, x_1, x_2, x_3] + \ldots$$

Divided diff table is

x	$f(x)$	1st div diff	2nd div diff	3rd div diff
4	-43			
		42		
7	83		16	
		122		1
9	327		24	
		242		
12	1053			

$$f(x) = -43 + (x-4)\times 42 + (x-4)(x-7)\times 16 + (x-4)(x-7)(x-9)\times 1$$

$$= -43 + x^3 - 7x + 4x^2 + 28$$

$$f(x) = x^3 - 4x^2 - 7x - 15$$

$\therefore f(10) = 1000 - 400 - 70 - 15$

$= 600 - 85$

$f(10) = 515$

pbm: 3 Given the values

x	5	7	11	13	17
y	150	392	1452	2366	5202

evaluate y_9 using (i) Lagrange's formula ii) Newton's divided diff formula.

Soln: i) Lagrange's formula.

Given : $x_0 = 5$, $x_1 = 7$, $x_2 = 11$, $x_3 = 13$, $x_4 = 17$

$y_0 = 150$, $y_1 = 392$, $y_2 = 1452$, $y_3 = 2366$,

$y_4 = 5202$

Put $x = 9$ in Lagrage's formula.

$$y_9 = \frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)} \times 150 + \frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)} \times 392$$

$$+ \frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-7)(11-13)(11-17)} \times 1452 + \frac{(9-5)(9-7)(9-13)(9-17)}{(13-5)(13-7)(13-11)(13-17)}$$

$$\times 2366 + \frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)} \times 5202$$

$= 1615.3334 - 805.3334$

$\therefore y_9 = 810$

ii) Newton's divided difference formula.

$y = f(x)$

$= f(x_0) + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)$

$[x_0, x_1, x_2] + \cdots$

Divided diff-table is

x	$f(x)$	1st div diff	2nd div diff	3rd div diff	4th div diff
5	150				
		121			
7	392		24		
		265		1	
11	1452		32		0
		457		1	
13	2366		42		
		709			
17	5202				

Take $x=9$ in above formula,

$f(9) = 150 + (9-5)(121) + (9-5)(9-7)(24) + (9-5)(9-7)(9-11)$

$f(9) = 150 + 484 + 192 - 16 = 810$

values 121, 24 & 1 in above calculation obtain from divided diff table.

Inverse Interpolation

Interchanging variables x & y in Lagrange's formula

$$x = \frac{(y-y_1)(y-y_2)\cdots(y-y_n)x_0}{(y_0-y_1)(y_0-y_2)(y_0-y_n)} + \cdots + \frac{(y-y_0)\cdots\cdots(y-y_{n-1})x_n}{(y_n-y_0)(y_n-y_1)\cdots(y_n-y_{n-1})}$$

Iterative method:

$$P = \frac{1}{\Delta y_0}\left[y_p - y_0 - \frac{P(P-1)}{2!}\Delta^2 y_0 - \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 - \cdots\right]$$

$P_1 = \frac{1}{\Delta y_0}(y_p - y_0)$; $P_2 = \frac{1}{\Delta y_0}\left(y_p - y_0 - \frac{P_1(P_1-1)}{2!}\Delta^2 y_0\right)$

$$P_3 = \frac{1}{\Delta y_0}\left[y_p - y_0 - \frac{P_2(P_2-1)}{2!}\Delta^2 y_0 - \frac{P_2(P_2-1)(P_2-2)}{3!}\Delta^3 y_0\right]$$

Pbm: 1

Soln: To find x, when $y=7$

U.K Lagrange's inverse interpolation formula

Here $x_0 = 1$, $x_1 = 3$, $x_2 = 4$

$y_0 = 4$, $y_1 = 12$, $y_2 = 19$

$$x = \frac{(y-y_1)(y-y_2)}{(y_0-y_1)(y_0-y_2)} \times x_0 + \frac{(y-y_0)(y-y_2)}{(y_1-y_0)(y_1-y_2)} \times x_1 + \frac{(y-y_0)(y-y_1)}{(y_2-y_0)(y_2-y_1)} \times x_2$$

$$= \frac{(7-12)(7-19)}{(4-12)(4-19)} \times 1 + \frac{(7-4)(7-19)}{(12-4)(12-19)} \times 3 + \frac{(7-4)(7-12)\times 4}{(19-4)(19-12)}$$

$$= \frac{(-5)(-11)}{(-8)(-15)} + \frac{3(-12)}{8(-7)} \times 3 + \frac{3(-5)\times 4}{(15)(7)}$$

$$= 0.5 + 1.9286 - 0.5714$$

$$x = 1.8572$$

Pbm: 2

Tabulate $y = x^3$ for $x = 2, 3, 4, 5$ & calculate the cube root of 10 correct to 3 decimal places.

Soln:

For $x = 2, 3, 4, 5$ the values of y are 8, 27, 64, 125 respectively $(y = x^3)$.

Here $h = 1$,

Form forward difference table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
2	8			
		19		
3	27		18	
		37		6
4	64		24	
		61		
5	125			

First approximation is given by

$$P_1 = \frac{1}{\Delta y_0}(y_p - y_0)$$

$$= \frac{1}{19}(10-8)$$

$$= \frac{2}{19}$$

$$P_1 = 0.1$$

2nd approximation is

$$P_2 = \frac{1}{\Delta y_0}\left[y_p - y_0 - \frac{P_1(P_1-1)}{2!}\Delta^2 y_0\right]$$

$$= \frac{1}{19}\left[10 - 8 - \frac{(0.1)(0.1-1)}{2!}(18)\right]$$

$$= \frac{1}{19}\left[2 - \frac{(0.1)(-0.9)}{2}(18)\right]$$

$$= \frac{1}{19}\left[2 + 0.81\right] = \frac{2.81}{19}$$

$$P_2 = 0.15$$

3rd approximation is

$$P_3 = \frac{1}{\Delta y_0}\left[y_p - y_0 - \frac{P_2(P_2-1)}{2!}\Delta^2 y_0 - \frac{P_2(P_2-1)(P_2-2)}{3!}\Delta^3 y_0\right]$$

$$= \frac{1}{19}\left[10 - 8 - \frac{(0.15)(-0.85)}{2}\times 18 - \frac{(0.15)(-0.85)(-1.85)}{6}\times 6\right]$$

$$= \frac{1}{19}\left[2 + 1.1475 - 0.2359\right]$$

$$= \frac{2.9116}{19}$$

$$P_3 = 0.1532$$

4th approximation is

$$P_4 = \frac{1}{\Delta y_0}\left[y_p - y_0 - \frac{P_3(P_3-1)}{2!}\Delta^2 y_0 - \frac{P_3(P_3-1)(P_3-2)}{3!}\Delta^3 y_0\right]$$

$$= \frac{1}{19}\left[10 \cdot 8 - \frac{0.1532(-0.8468)}{2} \times 18 - \frac{0.1532(-0.8468)(-1.8468)}{6} \times 6\right]$$

$$= \frac{1}{19}\left[2 + 1.1676 - 0.2396\right]$$

$$= \frac{2.928}{19}$$

$$= 0.1542$$

Here $P_4 \approx P_5 = 0.154$ (correct to 3 decimal places)

We have to find $\sqrt[3]{10}$, since 10 lies between values of y corresponding to $x = 2$ & $x = 3$,

Required value of $\sqrt[3]{10}$ is $2 + hP_5$

$$\sqrt[3]{10} = 2 + 0.154$$

$$\sqrt[3]{10} = 2.154$$

Unit 4

Unit – IV

Matrices and Linear system of Equations – Gaussian Elimination method, Gauss – Jordan method, Modification of the Gauss method to compute the inverse – Method of Factorization – Iterative method – Jacobi and Gauss Seidal methods.

Simultaneous Equation: Unit - IV Remaining

Introduction :-

Simultaneous linear algebric equations occur in several engineering and statistical Problems. In this chapter we deal with several numerical methods for solving such system of equations

(1)

Simultaneous Equation:

A system of m linear Equation in n unknown $x_1, x_2, \ldots x_n$ is given by

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\cdots\cdots\cdots\cdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

This set of equations can be written in the matrix form $Ax = B$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

The $m \times n$ matrix A is called the co-efficient matrix

The $m \times (n+1)$ matrix given by

$$(A, B) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$ is called the

augmentedmatrix of the system.

A given system of linear equations can be solved by Cramer's rule or by matrix methods. But this method becomes tedious for large system. Hence we develop several numerical methods

(2)

i, Gauss elimation method

ii, Gauss Jordan method

iii, Crout's Method

iv, Seidal method for finding the solutions which

v, Gauss Jacobi

are well suited for implementation in computers.

Some methods yield the required result after some computations which can be specified in advance. Such method are called direct methods

An iterative method is one in which we start from an approximation to the actual solution and obtain better and better approximation from a computational cycle which is repeated sufficently many number of times to the get solution to the desired accuracy

Back substitution:

Consider a system of simultaneous linear equations given by $AX=B$, where A is an $n\times n$ co-efficient matrix.

Suppose the matrix A is upper triangular

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \cdots & a_{1n} \\ 0 & a_{22} & a_{23} \cdots & a_{2n} \\ 0 & 0 & a_{33} \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & a_{nn} \end{pmatrix}$ then the given system takes the form

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ 0 & 0 & a_{33} \cdots & a_{3n} \\ \vdots & & & \\ 0 & 0 & 0 \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

ie) $a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$

$a_{22}x_2 + \cdots\cdots + a_{2n}x_n = b_2$

$\cdots\cdots\cdots$

$a_{n-1\,n-1}x_{n-1} + a_{n-1\,n}x_n = b_{n-1}$

$a_{nn}x_n = b_n$ (3)

From the last equation, we get $x_n = \frac{b_n}{a_{nn}}$

substituting the values of x_n in the previous equation we get

$$x_{n-1} = \frac{1}{a_{n-1\,n-1}}\left[b_{n-1} - a_{n-1}\left(\frac{b_n}{a_n}\right)\right]$$

Proceeding like this, we can find all x_i's. This procedure is known as backward substitution

similarly considering lower triangular matrix

$$A = \begin{pmatrix} a_{11} & 0, & 0 \cdots & 0 \\ a_{12} & a_{22} & 0 \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{m} \end{pmatrix}$$

the given system takes the form

$$a_{11}x_1 = b_1$$

$$a_{21}x + a_{22}x_2 = b_2$$

$$\cdots\cdots$$

$$a_{n1}x_1 + \cdots + a_{nn}x_n = b_n$$

From the first equation we get

$x_1 = \frac{b_1}{a_{11}}$, sub the value x_1 in the next equ. we get

$$x_2 = \frac{1}{a_{22}}\left[b_2 - a_{21}\left(\frac{b_1}{a_{11}}\right)\right]$$

proceeding like this we can find all x_i's. this produce is known as forward substitution.

Example:

Solve the following equation by back substitution

$2x + 3y + 2z = 7$, $-3y + z = -2$, $4z = 4$

soln:

$$2x + 3y + 2z = 7 \dashrightarrow (1)$$

$$-3y + z = -2 \dashrightarrow (2)$$

$$4z = 2 \dashrightarrow (3)$$

From the last equation $z=1$, sub $z=1$ in ②

we get $-3y=-3$ Hence $y=1$ sub $z=1$ and $y=1$ in ①

we get $2x+3+2=7$

Hence $x=1$

④

$\therefore x=y=z=\underline{1}$

Gauss Elimination Method:-

Gauss Elimination method is a direct method which consists of transforming the given system of simultaneous equation to an upper triangular system. From this transformed system the required solution can be obtained by the method of back substitution

Consider the system of n equations in n unknowns given by $AX=B$ where A is the co-efficient matrix

The Augmented Matrix is

$$(A,B)=\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right)$$

To transform the system to an equivalent upper triangular system. We use the following row operations.

I. the row operation $R_i \longrightarrow R_i - \frac{a_{i1}}{a_{11}} R_1$ $i=1,2,3,\ldots n$ makes all the entries $a_{21}, a_{31} \ldots a_{n1}$ in the first column zero

Here the first equations is the pivotal equation $a_{11} \neq 0$ is called pivot and $\frac{-a_{i1}}{a_{11}}$ for $i = 2, 3,$ are called multiples for first elimination. If $a_{11} = 0$ we interchange the first row with another suitable row so as to have $a_{11} \neq 0$

5

II) Next we do the row operation $R_i \to R_i - \frac{a_{2i}}{a_{22}} R_2;$ $i = 3, 4, \ldots n$. This makes all entries $a_{32}, a_{42}, \ldots a_{n2}$ in the second column zero

III) In general the row operation

$$R_i \to R_{i-1} - \frac{a_{ik}}{a_{kk}} R_k; \quad i = k+1; k+2 \ldots \ldots n$$

will make all the entries $a_{k+1\,k}, a_{k+2\,k}, \ldots a_{nk}$ in the k^{th} column zero

Hence the given system of equations is reduced to the form $UX = D$ where U is an upper triangular matrix. The required solution by the method of Back substitution

Problem:-1 Solve the equations $x + y = 2$ and $2x + 3y = 5$ by Gauss elimination method.

Soln: The given set of equation can be written as

$$\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

The augmented matrix is

$$(A, B) = \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 3 & 5 \end{array} \right)$$

We note $a_{11} = 1 \neq 0$ is the pivot. The first equation is the pivot equation $-\left(\frac{2}{1}\right)$ is the multiplier for the second equation

$$(A, B) \sim \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \end{array} \right) \quad \begin{array}{l} R_1 \to R_1 \\ R_2 \to R_2 - 2R_1 \end{array}$$

The given set of equation is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$x+y=2$ and $y=1$

By back substitution we get $y=1$ and $x=1$

2) Solve the following system of equation using Gaussian elimination method.

$$x+y+z=9$$
$$2x-3y+4z=13$$
$$3x+4y+5z=40$$

(6)

Soln: The given set of equation can be written as

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & 4 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 13 \\ 40 \end{pmatrix}$$

The agumented matrix is $(A,B)=$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 2 & -3 & 4 & 13 \\ 3 & 4 & 5 & 40 \end{array}\right)$$

$$(A,B) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -5 & 2 & -5 \\ 0 & 1 & 2 & 13 \end{array}\right) \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & -5 & 2 \\ 0 & 0 & 12/5 \end{pmatrix} \quad R_3 \to R_3 + \frac{1}{5} R_2$$

The given system of equation reduces to the system.

$$12/5 \; z = 12$$
$$-5y + 2z = -5$$
$$\text{and } x+y+z=9$$

Now, by back substitution we obtain the solution $x=1$, $y=3$, $z=5$

Gauss - Jordan elimination method.

Consider the system of equation $AX=B$

If A is a diagonal matrix the given system reduces to

$$⑦ \quad \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

This system reduces to the following n equations

$$a_{11}x_1 = b_1 \; ; \quad a_{22}x_2 = b_2 \, , \quad a_{nn}x_n = b_n$$

Hence we get the solution directly as $x_1 = \frac{b_1}{a_{11}}$, $x_2 = \frac{b_2}{a_{22}} \cdots x_n \frac{b_n}{a_{nn}}$

The method of the solution directly as $x_1 = \frac{b_1}{a_{11}}, x_2 = \frac{b_2}{a_{22}} \cdots x_n = \frac{b_n}{a_{nn}}$. the method of obtaining the solution of the system of equation by reducing the matrix A to a diagonal matrix is known as Gauss-Jordon elimination method.

Problem:- 1

Solve the following equations by Gauss Jordon method.

$x+y=2 \; , 2x+3y=5$

soln:

The augmented matrix is $(A, B) = \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 3 & 5 \end{array}\right)$

$$(A,B) \sim \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \end{array}\right) \quad R_2 \rightarrow R_2 - 2R_1$$

$$\sim \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right) \quad R_1 \rightarrow R_1 - R_2$$

$$\sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$x = 1 \, , \; y = 1$$

Pablm:2 Solve the following system of equations by Gauss Jordon method. $x+y+z=9, \quad 2x-3y+4z=13, \quad 3x+4y+5z=40$

soln:

The augmented matrix is

$$(A, B) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 2 & -3 & 4 & 13 \\ 3 & 4 & 5 & 40 \end{array}\right)$$

$$(A,B) \sim \begin{pmatrix} 1 & 1 & 1 & | & 9 \\ 0 & -5 & 2 & | & -5 \\ 0 & 1 & 2 & | & 13 \end{pmatrix} \begin{matrix} R_2 \to R_2 + (-2)R_1 \\ R_3 \to R_3 + (-3)R_1 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 7/5 & | & 8 \\ 0 & -5 & 2 & | & -5 \\ 0 & 0 & 12/5 & | & 12 \end{pmatrix} \begin{matrix} R_1 \to R_1 + 1/5\, R_2 \\ R_3 \to R_3 + 1/5\, R_2 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & | & 1 \\ 0 & -5 & 0 & | & -15 \\ 0 & 0 & 12/5 & | & 12 \end{pmatrix} \begin{matrix} R_1 \to R_1 + \left(-\frac{7}{12}\right) R_3 \\ R_2 \to R_2 + \left(-\frac{10}{12}\right) R_3 \end{matrix}$$

(8) The given system reduces to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 12/5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -15 \\ 12 \end{pmatrix}$$

$x = 1$, $-5y = -15$, $12/5\, z = +12$

Hence $x = 1$, $y = 3$, $z = 5$

Prblm: 3

Solve the following system of equations by Gauss - Jordon method. $5x - 2y + 3z = 18$, $x + 7y - 3z = -22$, $2x - y + 6z = 22$

soln: The augmented matrix is

$$(A,B) = \begin{pmatrix} 5 & -2 & 3 & | & 18 \\ 1 & 7 & -3 & | & -22 \\ 2 & -1 & 6 & | & 22 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 7 & -3 & | & -22 \\ 5 & -2 & 3 & | & 18 \\ 2 & -1 & 6 & | & 22 \end{pmatrix} R_1 \leftrightarrow R_2$$

$$\sim \begin{pmatrix} 1 & 7 & -3 & | & -22 \\ 0 & -37 & 18 & | & 128 \\ 0 & -15 & 12 & | & 66 \end{pmatrix} \begin{matrix} R_2 \to R_2 - 5R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 7 & -3 & | & -22 \\ 0 & 1 & \frac{-18}{37} & | & \frac{128}{37} \\ 0 & -15 & 12 & | & 66 \end{pmatrix} R_2 \to \frac{-1}{37} R_2$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 15/37 & 82/37 \\ 0 & 1 & -18/37 & -128/37 \\ 0 & 0 & 174/37 & 522/37 \end{array}\right) \begin{array}{l} R_1 \to R_1 - 7R_2 \\ \\ R_3 \to R_3 + 15R_2 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 15/37 & 82/37 \\ 0 & 1 & -18/37 & -128/37 \\ 0 & 0 & 1 & 3 \end{array}\right) R_3 \to \frac{37}{174} R_3$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right) \begin{array}{l} R_1 \to R_1 - \frac{15}{37} R_3 \\ R_2 \to R_2 + \frac{18}{37} R_3 \end{array}$$

The system of equation reduces to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

$x = 1$, $y = -2$, $z = 3$

Calculation of Inverse of a matrix:-

Let A be a $n \times n$ non-singular matrix

let $X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix}$

be the inverse of A

$\therefore AX = I$, where I is the unit matrix of order n

$\therefore AX = I$ gives

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

The equ is equivalent to the following n system of simultaneous equations

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

$$\cdots \quad \cdots \quad \cdots$$

10

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

Each of the system of the above n system of equation can be solved by Gauss Elimination or Gauss Jordon method.

Problm:-1

Find the inverse of the matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$ using Gaussian method.

soln:- Let $X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$ be the inverse of A.

$\therefore AX = I_3$ where I_3 is the 3×3 Identity matrix

The agumented system A can be written as

$$A = \left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 3 & 0 & 1 & 0 \\ 1 & 4 & 9 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1/2 & 3/2 & -3/2 & 1 & 0 \\ 0 & 7/2 & 17/2 & -1/2 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_2 \to R_2 - 3/2\, R_1 \\ R_3 \to R_3 - 1/2\, R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1/2 & 3/2 & -\frac{3}{2} & 1 & 0 \\ 0 & 0 & -2 & 10 & -7 & 1 \end{array}\right) \quad R_3 \to R_3 - 7R_2$$

The equation $AX = I_3$ is the equivalent to the following three systems

$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1/3 & 3/2 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \end{pmatrix} = \begin{pmatrix} 1 \\ -3/2 \\ 10 \end{pmatrix} \rightarrow (1)$$

$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1/2 & 3/2 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ +1 \\ -7 \end{pmatrix} \rightarrow (2)$$

(11)

$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1/2 & 3/2 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_{13} \\ x_{23} \\ x_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rightarrow (3)$$

From (1) we get

$$2x_{11} + x_{21} + x_{31} = 1$$

$$1/2\, x_{21} + \frac{3}{2} x_{31} = -3/2$$

$$-2x_{31} = 10$$

By Backward substitution we get,

$$x_{31} = -5 \quad , x_{21} = 12 \quad ; x_{11} = \cancel{8}\ -3$$

$\hookrightarrow (4)$

From (2) we get

$$2x_{12} + x_{22} + x_{32} = 0$$

$$\frac{1}{2} x_{22} + \frac{3}{2} x_2 = 1$$

$$-2x_{32} = -7$$

By Backward Substitution, we get

$$x_{32} = 7/2 \ ; \ x_{22} = \frac{-17}{2} ,$$

$$x_{32} = \frac{5}{2} \rightarrow (5)$$

From (3) we get

$$2x_{13} + x_{23} + x_{33} = 0$$

$$1/2\, x_{23} + 3/2\, x_{33} = 0$$

$$-2x_{33} = 1$$

By Backward substitution, we get

$$x_{33} = -1/2 \ , \ x_{23} = 3/2$$

$$x_{13} = -1/2 \rightarrow (6)$$

From ④,⑤,⑥, we get the inverse of A as,

$$X = \begin{pmatrix} -3 & 5/2 & -1/2 \\ 12 & -17/2 & 3/2 \\ -5 & 7/2 & -1/2 \end{pmatrix}$$

Crout's Method:

Consider the system of n equations given by $AX = B$

where $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$ and $B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$

Crout's method of solving ① is a direct method which involves the following steps.

12

Step: 1

The matrix A is expressed in the form

$$A = LU \longrightarrow ②$$

where L is a lower triangular matrix given by

$$L = \begin{pmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ \vdots & & & & \\ l_{n1} & l_{n2} & \cdots & & l_{nn} \end{pmatrix}$$

and U is an upper triangular matrix given by.

$$U = \begin{pmatrix} 1 & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & 1 & u_{23} & \cdots & u_{2n} \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

Step: 2

From $A = LU$, L and U can be obtained by equating the corresponding elements of matrix on the both sides

Step: 3

Now, using ② in ①, we get $LUX = B$

Let $UX = B'$ where $B' = \begin{pmatrix} b'_1 \\ b'_2 \\ \vdots \\ b'_n \end{pmatrix}$

Then ③ becomes $LB' = B$

Step: 4

From $LB' = B$ we can obtain B' by forward substitution

Step: 5

From $UX = B'$, X can be found by backward substitution.

Remark:

Crout's method can be used to find the inverse of a non-singular matrix.

since $A = LU$, we get $A^{-1} = U^{-1}L^{-1}$

Prblm :- 1

Solve the following equation by Crout's method

$x + y = 2$, $2x + 3y = 5$

13

Soln:

The given system is $AX = B$

where $A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$; $X = \begin{pmatrix} x \\ y \end{pmatrix}$, $B = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \rightarrow ①$

Let $A = LU$

where $L = \begin{pmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{pmatrix}$ and $U = \begin{pmatrix} 1 & u_{12} \\ 0 & 1 \end{pmatrix} \rightarrow ②$

$A = LU$

$\therefore \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} l_{11} & l_{11}u_{12} \\ l_{21} & l_{21}u_{12} + l_{22} \end{pmatrix}$

$\therefore l_{11} = 1$, $l_{11}u_{12} = 1 \rightarrow u_{12} = 1$

$\therefore l_{21} = 2$, $l_{21}u_{12} + l_{22} = 3 \Rightarrow l_{22} = 1$

$\therefore L = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ and $U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

sub $A = LU$ in ① we get $LUX = B \rightarrow ③$

Let $UX = B'$ where $B' = \begin{pmatrix} b'_1 \\ b'_2 \end{pmatrix}$

$LB' = B$

$\therefore \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} b'_1 \\ b'_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

$b_1' = 2$ and $2b_1' + b_2' = 5$ Hence $B' = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$b_2' = 1$

Now, $UX = B' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\therefore x + y = 2$ and $y = 1$

$\therefore x = 1$ & $y = 1$.

Prblm : 2

Solve the following equations by Crout's method $x + y + z = 9$, $2x - 3y + 4z = 13$; $3x + 4y + 5z = 40$

soln:-

The given sets of equations can be written as, $AX = B \rightarrow (1)$.

14

where $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$; $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$; $B = \begin{pmatrix} 9 \\ 13 \\ 40 \end{pmatrix}$

Let $A = LU$

where $L = \begin{pmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{pmatrix}$ and

$U = \begin{pmatrix} 1 & U_{12} & U_{13} \\ 0 & 1 & U_{23} \\ 0 & 0 & 1 \end{pmatrix}$

$A = LU$ gives $\begin{pmatrix} L_{11} & L_{11}U_{12} & L_{11}U_{13} \\ L_{21} & L_{21}U_{12} + l_{22} & L_{21}U_{13} + l_{22}U_{23} \\ L_{31} & L_{31}U_{12} + l_{32} & L_{31}U_{13} + l_{32}U_{23} + l_{33} \end{pmatrix}$

$= \begin{pmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$

Equating the elements of a matrix and we have

$= 40$

Column 1	Column 2	Column 3
$l_{11} = 1$	$l_{11} u_{12} = 1$ $\Rightarrow u_{12} = 1$	$l_{11} u_{13} = 1$ $\Rightarrow u_{13} = 1$
$l_{21} = 2$	$l_{21} u_{12} + l_{22} = -3$ $\Rightarrow 2 + l_{22} = -3$ $l_{22} = -5$	$l_{21} u_{13} + l_{22} u_{23} = 4$ $\Rightarrow 2 - 5u_{23} = 4$ $u_{23} = -2/5$
$l_{31} = 3$	$l_{31} u_{12} + l_{32} = 4$ $\Rightarrow 3 + l_{32} = 4$ $l_{32} = 1$	$l_{31} u_{13} + l_{32} u_{23} + l_{33} = 5$ $\Rightarrow 3 - \frac{2}{5} + l_{33} = 5$ $\Rightarrow l_{33} = 12/5$

$$\therefore L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -5 & 0 \\ 3 & 1 & -2/5 \end{pmatrix} \text{ and } U = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -2/5 \\ 0 & 0 & 1 \end{pmatrix}$$

Now substituting $A = LU$ in ① we get

$$LUx = B \longrightarrow ③$$

15

Let $Ux = B'$ where $B' = \begin{pmatrix} b_1' \\ b_2' \\ b_3' \end{pmatrix}$

$$\therefore LB' = B \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 2 & -5 & 0 \\ 3 & 1 & 12/5 \end{pmatrix} \begin{pmatrix} b_1' \\ b_2' \\ b_3' \end{pmatrix} = \begin{pmatrix} 9 \\ 13 \\ 40 \end{pmatrix}$$

$$b_1' = 9$$

$$2b_1' - 5b_2' = 13$$

$$\Rightarrow 18 - 5b_2' = 13$$

$$\Rightarrow b_2' = 1$$

$$3b_1' + b_2' + \frac{12}{5} b_3' = 40$$

$$\Rightarrow 27 + 1 + 12/5 \; b_3' = 40$$

$$\Rightarrow b_3' = 5$$

$$B' = \begin{pmatrix} 9 \\ 1 \\ 5 \end{pmatrix}$$

Now, $Ux = B' \Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -2/5 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 1 \\ 5 \end{pmatrix}$

$$\Rightarrow x + y + z = 9$$

$$y - 2/5 \, z = 1$$

$$z = 5$$

By Back substitution we get $z = 5$

$y=3$, $x=1$. Hence the required solution is $x=1$, $y=3$, $z=5$

3 Solve the following system of equation by Crout's equation method $2x-6y+8z=24$; $5x+4y-3z=2$; $3x+y+2z=16$

soln:

The given system of equation can be written as $Ax=B \rightarrow (1)$

where $A=\begin{pmatrix} 2 & -6 & 8 \\ 5 & 4 & -3 \\ 3 & 1 & 2 \end{pmatrix}$; $x=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $B=\begin{pmatrix} 24 \\ 2 \\ 16 \end{pmatrix}$

Let $A=LU$ where $L=\begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}$ and

$U=\begin{pmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$.

16

$A=LU$ gives

$$\begin{pmatrix} 2 & -6 & 8 \\ 5 & 4 & -3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12}+l_{22} & l_{21}u_{13}+l_{22}u_{23} \\ l_{31} & l_{31}u_{12}+l_{32} & l_{31}u_{13}+l_{32}u_{23}+l_{33} \end{pmatrix} = \begin{pmatrix} 2 & -6 & 8 \\ 5 & 4 & -3 \\ 3 & 1 & 2 \end{pmatrix}$$

Equating the elements of the matrix we have.

Column 1	Column 2	Column 3.
$l_{11} = 2$	$l_{11} U_{12} = -6$ $\Rightarrow 2U_{12} = -6$ $U_{12} = -3$	$l_{11} U_{13} = 8$ $\Rightarrow 2U_{13} = 8$ $\Rightarrow U_{13}\ 4$
$l_{21} = 5$	$l_{21} U_{12} + l_{22} = 4$ $\Rightarrow 5(-3) + l_{22} = 4$ $\Rightarrow l_{22} = 19$	$l_{21} U_{13} + l_{22} U_{23} = -3$ $\Rightarrow 5 \times 4 + 19 U_{23} = -3$ $\Rightarrow U_{23} = -23/19$
$l_{31} = 3$	$l_{31} U_{12} + l_{32} = 1$ $\Rightarrow 3(-3) + l_{32} = 1$ $\Rightarrow l_{32} = 10$	$l_{31} U_{13} + l_{32} U_{23} + l_{33} = 2$ $\Rightarrow 3 \times 4 + 10\left(\frac{-23}{19}\right) + l_{33} = 2$ $\Rightarrow l_{33} = 40/19$

$$\therefore L = \begin{pmatrix} 2 & 0 & 0 \\ 5 & 19 & 0 \\ 3 & 10 & \frac{40}{19} \end{pmatrix} \text{ and } U = \begin{pmatrix} 1 & -3 & 4 \\ 0 & 1 & -23/19 \\ 0 & 0 & 1 \end{pmatrix}$$

Now, sub $A = LU$ in ① we get

17

$$LUX = B \rightarrow ②$$

Let $UX = B'$ where $B' = \begin{pmatrix} b'_1 \\ b'_2 \\ b'_3 \end{pmatrix}$

$$LB' = B \Rightarrow \begin{pmatrix} 2 & 0 & 0 \\ 5 & 19 & 0 \\ 3 & 10 & \frac{40}{19} \end{pmatrix} \begin{pmatrix} b'_1 \\ b'_2 \\ b'_3 \end{pmatrix} = \begin{pmatrix} 24 \\ 2 \\ 16 \end{pmatrix}$$

$$2b'_1 = 24$$
$$\Rightarrow b'_1 = 12$$

$$5b'_1 + 19 b'_2 = 2$$
$$60 + 19 b'_2 = 2$$
$$\Rightarrow b'_2 = \frac{-58}{19}$$

$$3b'_1 + 10\left(\frac{-58}{19}\right) + \frac{40}{19} b'_3 = 16$$
$$\Rightarrow 36 - \frac{580}{19} + \frac{40}{19} b'_3 = 16$$
$$\Rightarrow b'_3 = 5$$

$$\therefore B' = \begin{pmatrix} 12 \\ -58/19 \\ 5 \end{pmatrix}$$

$$\text{Now, } UX = B' \Rightarrow \begin{pmatrix} 1 & -3 & 4 \\ 0 & 1 & -23/19 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12 \\ -58/19 \\ 5 \end{pmatrix}$$

$\Rightarrow x - 3y + 4z = 12$

$$y - \frac{23}{19} z = \frac{-58}{19}$$

$$z = 5$$

By Back substitution, we get $z = 5$, $y = 3$, $x = 1$

Hence the required solution is $x = 1$, $y = 3$, $z = 5$

4. Find the inverse of a matrix by Crout's method

$$\begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$$

soln: Let $A = LU$, where L is the lower triangular matrix

$$L = \begin{pmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{pmatrix}$$ and U is an upper triangular matrix

$$U = \begin{pmatrix} 1 & U_{12} & U_{13} \\ 0 & 1 & U_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

(18)

Now, we find L and U from $LU = A$ as follows

$$\begin{pmatrix} L_{11} & 0 & 0 \\ L_{21} & l_{22} & 0 \\ L_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} 1 & U_{12} & U_{13} \\ 0 & 1 & U_{23} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$$

$$\begin{pmatrix} L_{11} & L_{11}U_{12} & l_{11}U_{13} \\ L_{21} & l_{21}U_{12} + l_{22} & l_{21}U_{13} + l_{22}U_{23} \\ L_{31} & l_{31}U_{12} + l_{32} & l_{31}U_{13} + l_{32}U_{23} + l_{33} \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$$

we equate elements of the matrix.

Column 1	Column 2	Column 3
$l_{11} = 2$	$l_{11} U_{12} = 1$ $\Rightarrow U_{12} = 1/2$	$l_{11} U_{13} = 1$ $\Rightarrow U_{13} = 1/2$
$l_{21} = 3$	$l_{12} U_{12} + l_{22} = 2$ $\Rightarrow 3 \times \frac{1}{2} + l_{22} = 2$ $\Rightarrow l_{22} = 1/2$	$l_{21} U_{13} + l_{22} U_{23} = 3$ $\Rightarrow 3/2 + 1/2\ U_{23} = 3$ $\Rightarrow U_{23} = 3$
$l_{31} = 1$	$l_{31} U_{12} + l_{32} = 4$ $\Rightarrow 1 \times \frac{1}{2} + l_{32} = 4$ $\Rightarrow l_{32} = 7/2$	$l_{31} U_{13} + l_{32} U_{23} + l_{33} = 9$ $1 \times 1/2 + 7/2 \times 3 + l_{33} = 9$ $\Rightarrow l_{33} = -2.$

$\therefore L = \begin{pmatrix} 2 & 0 & 0 \\ 3 & 1/2 & 0 \\ 1 & 7/2 & -2 \end{pmatrix}$ and $U = \begin{pmatrix} 1 & 1/2 & 1/2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$

Let $L^{-1} = \begin{pmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$ (19)

$\because$ (Inverse of a lower triangular matrix is a lower triangular)

$\therefore LL^{-1} = I$

$$\begin{pmatrix} 2 & 0 & 0 \\ 3 & 1/2 & 0 \\ 1 & 7/2 & -2 \end{pmatrix} \begin{pmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\therefore\ 2x_{11} = 1$; $3x_{11} + 1/2\ x_{21} = 0$, $1/2\ x_{22} = 1$

$x_{11} + 7/2\ x_{21} - 2x_{31} = 0$; $7/2\ x_{22} - 2x_{32} = 0$ and

$-2x_{33} = 1$ Hence $x_{11} = 1/2$, $x_{21} = -3$, $x_{22} = 2$,

$x_{31} = 5$, $x_{32} = 7/2$ and $x_{33} = -1/2$

$\therefore L^{-1} \begin{pmatrix} 1/2 & 0 & 0 \\ -3 & 2 & 0 \\ -5 & 7/2 & -1/2 \end{pmatrix}$

Now, let $U^{-1} = \begin{pmatrix} 1 & y_{12} & y_{13} \\ 0 & 1 & y_{23} \\ 0 & 0 & 1 \end{pmatrix}$

$UU^{-1} = I$

$$\Rightarrow \begin{pmatrix} 1 & 1/2 & 1/2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y_{12} & y_{13} \\ 0 & 1 & y_{23} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\therefore y_{12} + \frac{1}{2} = 0 \;;\; y_{23} + 3 = 0 \,,\, y_{13} + \frac{1}{2} y_{23} + \frac{1}{2} = 0$

$\therefore y_{12} = -\frac{1}{2} \;;\; y_{23} = -3$ and $y_{13} = 1$

$$U^{-1} = \begin{pmatrix} 1 & -1/2 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$$

Now, $A = LU$. Hence $A^{-1} = (LU)^{-1}$

20

$$A^{-1} = \begin{pmatrix} 1 & -1/2 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/2 & 0 & 0 \\ -3 & 2 & 0 \\ -5 & 7/2 & -1/2 \end{pmatrix} =$$

$$\begin{pmatrix} -3 & 5/2 & -1/2 \\ 12 & -17/2 & 3/2 \\ -5 & 7/2 & -1/2 \end{pmatrix}$$

Iterative methods Gauss Jacobi iteration method.

Defn: An $n \times n$ matrix A is said to be diagonally dominant if the absolute value of each leading diagonal elements is greater than or equal to the sum of the absolute values of the remaining elements in that row

For ex: $A = \begin{pmatrix} 10 & -5 & -2 \\ 4 & -10 & 3 \\ 1 & 6 & 10 \end{pmatrix}$ is a diagonally dominant matrix and $B = \begin{pmatrix} 2 & 3 & -1 \\ 5 & 8 & -4 \\ 1 & 1 & 1 \end{pmatrix}$ is not a diagonally dominant matrix

Note: In the system of simultaneous linear equation in n unknowns $AX = B$ if A is diagonally dominant then the system is said to be diagonal system

Thus the system of equations

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$ is a

diagonal system if

$$|a_1| \geq |b_1| + |c_1|$$

21

$$|b_2| \geq |a_2| + |c_2| \text{ and}$$
$$|c_3| \geq |a_3| + |b_3|.$$

The process of iteration in solving $AX=B$ will converge quickly if the co-efficient matrix A is a diagonally dominant. If the co-efficient matrix is not diagonally dominant. we must rearrange the equations in such a way. that the resulting co-efficient matrix. becomes dominant if possible before we apply the iteration method

we now discuss Gauss iteration method.

Consider the system of equations

$$a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n = c_1$$
$$a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n = c_2$$
$$\dots\dots\dots\dots$$
$$a_{n1} x_1 + a_{n2} x_2 + \dots\dots + a_{nn} x_n = c_n.$$

we assume that the co-efficient matrix of this system is diagonally dominant. the above equations can be rewritten as

$$x_1 = \frac{1}{a_{11}} (c_1 - a_{12} x_2 - a_{13} x_3 - \dots a_{1n} x_n) \quad \to (1)$$

$$x_2 = \frac{1}{a_{22}} (c_2 - a_{21} x_1 - a_{23} x_3 - \dots a_{2n} x_n) \quad \to (2)$$

$$\dots\dots\dots\dots$$

$$x_n = \frac{1}{a_{nn}} (c_n - a_{n1} x_1 - a_{n2} x_2 - \dots a_{nn} x_{n-1}) \quad \to (3)$$

we start with the initial values for the variables $x_1, x_2, x_3 \dots x_n$ to be $x_1(0), x_2^{(0)}, x_3^{(0)} \dots x_n^{(0)}$ using these values in (1) (2) ... (n) respectively we get $x_1^{(1)}, x_2^{(1)} \dots x_n^{(1)}$

putting $x_1 = x_1^{(1)}; x_2 = x_2^{(1)} \dots x_n = x_n^{(1)}$ in (1) (2) ... (n) respectively. we get the next approximations $x_1^{(2)}, x_2^{(2)} \dots x_n^{(2)}$. In general if the values of $x_1, x_2, \dots x_n$ in the r^{th} iteration are $x_1^{(r)}, x_2^{(r)} \dots x_n^{(r)}$, then $x_1^{(r+1)} = \frac{1}{a_{11}} [c_1 - a_{12} x_2^{(r)} - a_{13} x_3^{(r)} - \dots - a_{1n} x_n^{(r)}]$.

$$x_2^{(r+1)} = \frac{1}{a_{22}} [c_2 - a_{21} x_1^{(r)} - a_{23} x_3^{(r)} - \dots - a_{2n} x_n^{(r)}]$$

22

$$\dots \dots \dots$$

$$x_n^{(r+1)} = \frac{1}{a_{nn}} [c_n - a_{n1} x_1^{(r)} - a_{n2} x_2^{(r)} - \dots - a_{n\,n-1} x_{n-1}^{(r)}]$$

Note:

In solving a specific problem in the absence of any specific initial values for the variables we usually take the initial values of the variables to be zero

Prblm: 1

Check whether the system of equations $x + 6y - 2z = 5$; $4x + y + z = 6$; $-3x + y + 7z = 5$ is a diagonal system if not make it diagonal matrix

soln:

The co-efficient matrix

$$A = \begin{pmatrix} 1 & 6 & -2 \\ 4 & 1 & 1 \\ -3 & 1 & 7 \end{pmatrix}$$ is not a diagonally dominant as such. However effecting the operation $C_2 \leftrightarrow C_1$ (interchanging column one & two) we get

$$A \sim \begin{pmatrix} 6 & 1 & -2 \\ 1 & 4 & 1 \\ 1 & -3 & 7 \end{pmatrix}$$ which is

diagonally dominant matrix and the

system is a diagonal matrix system
The system of equations can be written as

$$\begin{pmatrix} 6 & 1 & -2 \\ 1 & 4 & 1 \\ 1 & -3 & 7 \end{pmatrix} \begin{pmatrix} y \\ x \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ 5 \end{pmatrix}$$

2) Is the system of equations diagonally dominant? If not make it diagonally dominant. $3x+9y-2z=10$, $4x+2y+13z=19$, $4x-2y+z=3$

soln:
Consider the co-efficient matrix

23

$$A = \begin{pmatrix} 3 & 9 & -2 \\ 4 & 2 & 13 \\ 4 & -2 & 1 \end{pmatrix}$$

Obviously, A is not diagonally dominant as it is

NOW, $A \sim \begin{pmatrix} 4 & 2 & 13 \\ 3 & 9 & -2 \\ 4 & -2 & 1 \end{pmatrix}$ $R_2 \leftrightarrow R_1$

$\sim \begin{pmatrix} 13 & 2 & 9 \\ -2 & 9 & 3 \\ 1 & -2 & 4 \end{pmatrix}$ $C_3 \leftrightarrow C_1$

This is a diagonally domiant matrix and the system of equations can be written as

$$\begin{pmatrix} 13 & 2 & 4 \\ -2 & 9 & 3 \\ 1 & -2 & 4 \end{pmatrix} \begin{pmatrix} z \\ y \\ x \end{pmatrix} = \begin{pmatrix} 19 \\ 10 \\ 3 \end{pmatrix}$$

3) Solve the following equation using Jacobi's iteration method $3x+4y+15z=54.8$; $x+12y+3z=39.66$, $10x+y-2z=7.74$

soln:
Co-efficient matrix of the given system of equation is $A = \begin{pmatrix} 3 & 4 & 15 \\ 1 & 12 & 3 \\ 10 & 1 & -2 \end{pmatrix}$

We note that A is not diagonally dominant

Also, $A \sim \begin{pmatrix} 10 & 1 & -2 \\ 1 & 12 & 3 \\ 3 & 4 & 15 \end{pmatrix}$ $R_1 \longleftrightarrow R_3$

which is diagonally dominant

The given system becomes

$$10x + y - 2z = 7.74 \rightarrow ①$$

$$x + 12y + 3z = 39.66 \rightarrow ②$$

$$3x + 4y + 15z = 54.8 \rightarrow ③$$

From ①, ②, ③ we get

24

$$x = \frac{1}{10}\left[7.74 - y + 2z\right] \rightarrow ④$$

$$y = \frac{1}{12}\left[39.66 - x - 3z\right] \rightarrow ⑤$$

$$z = \frac{1}{15}\left[54.8 - 3x - 4y\right] \rightarrow ⑥$$

First - Iteration

Let the initial value be. $x_0 = y_0 = z_0 = 0$

$$x_1 = \frac{1}{10}\left[7.74\right] = 0.774$$

$$y_1 = \frac{1}{12}\left[39.66\right] = 3.305$$

$$z_1 = \frac{1}{15}\left[54.8\right] = 3.6533$$

second iteration.

$$x_2 = \frac{1}{10}(7.74 - y_1 + 2z_1) = \frac{1}{10}(7.74 - 3.305 + 7.3066) = 1.1742$$

$$y_2 = \frac{1}{12}(39.66 - x_1 - 3z_1) = \frac{1}{12}(39.66 - 0.774 - 10.9599) = 2.3272$$

$$z_2 = \frac{1}{15}(54.8 - 3x_1 - 4y_1) = \frac{1}{15}(54.8 - 2.322 - 13.22) = 2.617$$

Third Iteration

$$x_3 = \frac{1}{10}(7.74 - y_2 + 2z_2) = \frac{1}{10}(7.74 - 2.3272 + 5.434)$$

$$= 1.0647$$

$$y_3 = \frac{1}{12}(39.66 - x_2 - 3z_2) = \frac{1}{12}(39.66 - 1.5142 - 7.856)$$

$$= 2.5529$$

$$z_3 = \frac{1}{15}(54.8 - 3x_2 - 4y_2) = \frac{1}{15}(54.8 - 3.5286 - 9.3088)$$

$$= 2.7979$$

Fourth iteration:-

$$x_4 = \frac{1}{10}[7.74 - y_3 + 2z_3] = \frac{1}{10}[7.74 - 2.5529 + 5.5958]$$

25

$$= 1.0783$$

$$y_4 = \frac{1}{12}[39.66 - x_3 - 3z_3] = \frac{1}{12}[39.66 - 1.0647 - 8.3937]$$

$$= 2.5168$$

$$z_4 = \frac{1}{15}[54.8 - 3x_3 - 4y_3]$$

$$= \frac{1}{15}[54.8 - 3.1941 - 10.2116]$$

$$= 2.7596$$

Fifth Iteration:

$$x_5 = \frac{1}{10}[7.74 - y_4 + 2z_4]$$

$$= \frac{1}{10}[7.74 + 2.5168 + 5.5192]$$

$$= 1.0742$$

$$y_5 = \frac{1}{12}[39.66 - x_4 - 3z_1]$$

$$= \frac{1}{12}[39.66 - 1.0788 - 8.2788]$$

$$= 2.5252$$

$$z_5 = \frac{1}{15}[54.8 - 3x_4 - 4y_4]$$

$= \frac{1}{15}[54.8 - 3.2349 - 10.0672]$

$= 2.7665$

Sixth iteration:

$x_6 = \frac{1}{10}[7.74 - y_5 + 2z_5]$

$= \frac{1}{10}[7.74 - 2.5252 + 5.533]$

$= 1.0748$

$y_6 = \frac{1}{12}[39.66 - x_5 - 3z_5]$

26

$= \frac{1}{12}[39.66 - 1.07142 - 8.2995]$

$= 2.5239$

$z_6 = \frac{1}{15}[54.8 - 3x_5 - 4z_5]$

$= \frac{1}{15}[54.8 - 3.226 - 10.1008]$

$= 2.7651$

Seventh iteration:

$x_7 = \frac{1}{10}[7.74 - y_6 + 2z_6]$

$= \frac{1}{10}[7.74 - 2.5239 + 5.5302]$

$= 1.0746$

$y_7 = \frac{1}{12}[39.66 - x_6 - 3z_6]$

$= \frac{1}{12}[39.66 - 1.0748 - 8.2953]$

$= 2.5242$

$z_7 = \frac{1}{15}[54.8 - 3x_6 - 4y_6]$

$= \frac{1}{15}[54.8 - 3.2244 - 10.0956]$

$= 2.7653$

After 7 iterations the difference in 6^{th} and 7^{th} iterations is very negligible.

Hence the solution of the system is given by $x = 1.075$, $y = 2.524$, $z = 2.765$ correct to three places of decimals.

Gauss - seidel iteration method :-

Gauss - seidel iteration method is a refinement of Gauss - Jacobi method

(27)

$$x_1 = \frac{1}{a_{11}} \left[c_1 - a_{12}x_2 - a_{13}x_3 - \cdots - a_{1n}x_n \right] \rightarrow (1)$$

$$x_2 = \frac{1}{a_{22}} \left[c_2 - a_{21}x_1 - a_{23}x_3 - \cdots - a_{2n}x_n \right] \rightarrow (2)$$

$$\vdots$$

$$x_n = \frac{1}{a_{nn}} \left[c_n - a_{n1}x_1 - a_{n2}x_2 - \cdots - a_{n,n-1}x_{n-1} \right] \rightarrow (n)$$

We start with the initial values.

$x_2^{(0)}, x_3^{(0)}, \ldots x_n^{(0)}$ and we get from (1)

$$x_1^{(1)} = \frac{1}{a_{11}} \left[c_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)} - \cdots - a_{1n}x_n^{(0)} \right]$$

In the second equation we us $x_1^{(1)}$ for x_1 and $x_3^{(0)}$ for x_3 etc ... and $x_n^{(0)}$ for x_n

(In the Jacobi method we use $x_1^{(0)}$ for x_1) Thus, we get

$$x_2^{(1)} = 1/a_{22} \left[c_2 - a_{11}x_1^{(1)} - a_{13}x_3^{(0)} - \cdots - a_{2n}x_n^{(0)} \right]$$

proceeding like this we find the first iteration values as

$$x_1^{(1)}, x_2^{(1)}, \ldots x_n^{(1)}$$

In general if the values of the variables in the r^{th} iteration are $x_1^{(r)}, x_2^{(r)}, \dots x_n^{(r)}$ then the values in the $(r+1)$ iteration are given by

$$x_1^{(r+1)} = \frac{1}{a_{11}}\left[c_1 - a_{12}x_2^{(r)} - a_{13}x_3^{(r)} - \dots - a_{1n}x_n^{(r)}\right]$$

$$x_2^{(r+1)} = \frac{1}{a_{22}}\left[c_2 - a_{21}x_1^{(r+1)} - a_{23}x_3^{(r)} - \dots - a_{2n}x_n^{(r)}\right]$$

$$\vdots$$

$$x_n^{(r+1)} = \frac{1}{a_{nn}}\left[c_n - a_{n1}x_1^{(r+1)} - a_{n2}x_2^{(r+1)} - \dots - a_{nn-1}x_{n-1}^{(r+1)}\right]$$

28

Problem :- 1

Solve $2x+y=3$, $2x+3y=5$ by Gauss seidel iteration method.

soln :-

Clearly the co-efficient matrix $\begin{pmatrix}2 & 1\\ 1 & 3\end{pmatrix}$ is diagonally dominant and hence Gauss seidel iteration method can be applied

The given equation can be written as

$$x = \tfrac{1}{2}(3-y) \longrightarrow ①$$

$$y = \tfrac{1}{3}(5-2x) \longrightarrow ②$$

putting $y=0$ in ①, we get $x=1.5$

using $x=1.5$ in ② we get $y=0.6667$

Putting $y=0.6667$ in ① we get $x=1.1667$

putting $x=1.1667$ in ② we get $y=0.8889$

putting $y=0.8889$ in ① we get $x=1.0556$

putting $x=1.0556$ in ② we get $y=0.9629$

putting $y=0.9629$ in ① we get $x=1.0186$

putting $x=1.0186$ in ② we get $y=0.9876$

putting $y=0.9876$ in ① we get $x=1.0068$

putting $x=1.0068$ in ② we get $y=0.9954$

using $y = 0.9959$ in ① we get $x = 1.0021$
using $x = 1.0021$ in ② we get $y = 0.9986$
using $y = 0.9986$ in ① we get $x = 1.0007$
putting $x = 1.0007$ in ② we get $y = 0.9995$
putting $y = 0.9995$ in ① we got $x = 1.0001$
putting $x = 1.0001$ in ② we get $y = 0.9999$
using $y = 0.9999$ in ① we get $x = 1$
using $x = 1$ in ② we get $y = 1$

Hence $x = 1, y = 1$ is the solution of the two equations.

29

Note: The above iteration can be simply carried out and exhibited in the following tabular form

Iteration	Start	1	2	3	4	5	6	7	8	9	10
x	–	1.5	1.6667	1.0556	1.0186	1.0062	1.0021	1.0007	1.0001	1	1
y	0	0.6667	0.8889	0.9629	0.9876	0.9959	0.9986	0.9995	0.9999	1	1

Problem: 2 solve the following system of equations using Gauss Seidel iteration method

$6x + 15y + 2z = 72$; $x + y + 54z = 110$,

$27x + 6y - z = 85$

Co-efficient matrix of the given system of equations is

$$A = \begin{pmatrix} 6 & 15 & 2 \\ 1 & 1 & 54 \\ 27 & 6 & -1 \end{pmatrix}$$

we note that A is not diagonally dominant. However it can be made diagonally dominant by changing rows as

$$A = \begin{pmatrix} 27 & 6 & -1 \\ 6 & 15 & 2 \\ 1 & 1 & 54 \end{pmatrix}$$ 30

Hence the corresponding system of equations is

$$27x + 6y - z = 85$$
$$6x + 15y + 2z = 72$$
$$x + y + 54z = 110$$

The above system of equations can be rewritten as

$$x = 1/27\,(85 - 6y + z) \rightarrow ①$$
$$y = 1/15\,(70 - 6x - 2z) \rightarrow ②$$
$$z = 1/54\,(110 - x - y) \rightarrow ③$$

<u>First iteration:</u>

putting $y=0$ and $z=0$ in ①

we get $x = 85/87 = 3.1481$

Putting $x = 3.1481$ and $z = 0$ in ② we get

$$y = 1/15\,[72 - 6 \times 3.1481] = 3.5408$$

Putting $x = 3.1481$ and $y = 3.5408$ in ③ we get

$$z = \frac{1}{54}\,[110 - 3.1481 - 3.5408]$$
$$= 1.9132$$

Second Iteration :-

putting $y = 3.5408$ and $z = 1.9132$ in ①

we get

$$x = 1/27\,[85 - 6\times 3.5408 + 1.9132]$$

$$= 2.4322$$

putting $x = 2.4322$ and $z = 1.9132$ in ② we get

$$y = \frac{1}{15}[72 - 6\times 2.4322 - 2\times 1.9132]$$

$$= 3.572$$

putting $x = 2.4328$ and $y = 3.572$ in ③ we get

$$z = \frac{1}{54}[110 - 2.4322 - 3.572]$$

$$= 1.9258$$

31

Third Iteration :-

putting $y = 3.572$ and $z = 1.9258$ in ①

we get,

$$x = 1/27\,[85 - 6\times 3.572 + 1.9258]$$

$$= 2.4257$$

putting $x = 2.4257$ and $z = 1.9258$ in ②

we get,

$$y = \frac{1}{15}[72 - 6\times 2.4257 - 2\times 1.9258]$$

$$= 3.5729$$

putting $x = 2.4257$ and $y = 3.5729$ in ①

we get

$$z = 1/54\,[110 - 2.4257 - 3.5729]$$

$$= 1.926$$

Fourth iteration:-

putting $y = 3.5729$ and $z = 1.926$ in ①

we get

$$x = \frac{1}{27}(85 - 6\times 3.5729 + 1.926)$$

$$= 2.4255$$

putting $x = 2.4255$ and $z = 1.926$ in ② we get

$$y = \frac{1}{15}(72 - 6\times 2.4255 - 2\times 1.926)$$

$$= 3.573$$

putting $x = 2.4255$ and $y = 3.573$ in ③

we get,

$$z = 1/54\,(110 - 2.4255 - 3.573)$$

$$= 1.926$$

The values of x, y, z in the third and fourth iteration are almost equal.

$\therefore$ The root of the system are $x = 2.425$, $y = 3.573$ and $z = 1.926$

Note:-

The above iteration can be simply carried out and shown in the following tabular form.

32

iteration	Initial values	1	2	3	4	5
x	—	3.1481	2.4322	2.4257	2.4255	2.4255
y	0	3.5408	3.5729	3.5729	3.5730	3.5730
z	0	1.9132	1.9258	1.9260	1.9260	1.9260

Problem:-3

Solve the following system of equations using Gauss seidel iteration $10x + 2y + z = 9$, $x + 10y - z = -22$, $-2x + 3y + 10z = 22$

Soln:- Clearly, the given system of equations is diagonally dominant. Hence it can be solved by Gauss-seidel iteration method.

The given system of equations can be written as

$$x = \frac{1}{10}(9 - 2y - z) \rightarrow ①$$

$$y = \frac{1}{10}(-22 - x + z) \rightarrow ②$$

$$z = \frac{1}{10}(22 + 2x - 3y) \rightarrow ③$$

First Iteration:

putting $y = 0$ and $z = 0$ in ① we get $x = 0.9$ putting $x = 0.9$ and $z = 0$ in ② we get

$$y = \frac{1}{10}(-22 - 0.9)$$

$$= -2.27$$

Putting $x = 0.9$ and $y = -2.29$ in (3) we get

$z = 1/10\,(22 + 2\times 0.9 - 3(-2.29)) = 3.067$

Second Iteration:-

putting $y = -2.29$ and $z = 3.067$ in (1)

we get

$x = 1/10\,[9 - 2(-2.29) - 3.067]$

$= 1.0513$

putting $x = 1.0513$ and $z = 3.067$ in (2)

we get

$y = 1/10\,(-22 - 1.0513 + 3.067]$

$= -1.9984$

33

putting $x = 1.0513$ and $y = -1.9984$ in (3)

we get

$z = 1/10\,[22 + 2\times 1.0513 - 3(-1.9984)]$

$= 3.0098$

Third iteration:

Putting $y = -1.9984$ and $x = 3.0098$ in (1)

$x = \frac{1}{10}[9 - 2(-1.9984) - 3.0098)$

-22 we get

$=$

putting $x = 0.9987$ and $z = 3.0098$ in (2)

we get

$y = \frac{1}{10}[-22 - 0.9987 + 3.0098]$

$= -1.9989$

putting $x = 0.9987$ and $y = -1.9989$ in (3)

we get

$z = \frac{1}{10}[22 + 2\times 0.9987 - 3(-1.9989)]$

$= 2.9994$

Fourth iteration:

putting $y = -1.9989$ and $z = 2.9994$ in

(1) we get

$x = \frac{1}{10}[9 - 2(-1.9989) - 2.9994]$

$= 0.9998$

putting $x = 0.9998$ and $z = 2.9994$ in

(2) we get

$y = \frac{1}{10}[-22 - 0.9998 + 2.9994]$

$= -2$

putting $x = 0.9998$ and $y = -2$ in (3) we

get

$z = \frac{1}{10}\left[22+2\times 0.9998 - 3(-2)\right]$

$= 3$

Proceeding like this in the next iteration

we get, $x = 1$, $y = -2$, $z = 3$

The roots are $x = 1$, $y = 2$, $z = 3$

4) Solve the following system of equations by i) Gauss - seidel method ii) Gauss Jacobi method. $28+4y-z=32$, $x+3y+10z=24$, $2x+17y+4z=35$

soln: The co-efficient matrix of the system is

$$A = \begin{pmatrix} 28 & 4 & -1 \\ 1 & 3 & 10 \\ 2 & 17 & 4 \end{pmatrix}$$ we find that A is not diagonally dominant. However

$$A \sim \begin{pmatrix} 28 & 4 & -1 \\ 2 & 17 & 4 \\ 1 & 3 & 10 \end{pmatrix} \quad R_2 \leftrightarrow R_3$$

which is diagonally dominant

Hence the given equations becomes

$$28x+4y-z=32 \longrightarrow ①$$
$$x+3y+10z=24 \longrightarrow ③$$
$$2x+17y+4z=35 \longrightarrow ②$$

From ① ② & ③ we have

$$x = \frac{1}{28}\left[32-4y+z\right] \longrightarrow ④$$
$$y = \frac{1}{17}\left[35-2x-4z\right] \longrightarrow ⑤$$
$$z = \frac{1}{10}\left[24-x-3y\right] \longrightarrow ⑥$$

i) Gauss seidal method:

First iteration:

Put $y=0$ and $z=0$ in ④ we get

$x = \frac{1}{28}[32] = 1.1429$

Put $x = 1.1429$, $z=0$ in ⑤ we get

$y = \frac{1}{17}[35-2.2858]$

$= 1.9244$

Put $y = 1.9244$, $x = 1.1429$ in (6) then

$$z = \frac{1}{10}[24 - 1.1429 - 5.7732]$$

$$= 1.7084$$

After first iteration, we have $x = 1.1429$

$y = 1.9244$, $z = 1.7084$

Second Iteration:-

Put $y = 1.9244$ and $z = 1.7084$ in (4)

$$x = \frac{1}{28}[32 - 7.6976 + 1.7084] = 0.9290$$

Put $x = 0.9290$ and $z = 1.7084$ in (5)

$$y = \frac{1}{17}[35 - 1.858 - 6.8336]$$

$$= 1.5476$$

35

Put $x = 0.9290$, and $y = 1.5476$ in (6)

$$z = \frac{1}{10}[24 - 0.9290 - 3(1.5476)]$$

$$= 1.8428$$

After second iteration, we have $x = 0.929$

$y = 1.5476$, $z = 1.8428$

Third iteration:-

Put $y = 1.5476$ and $z = 1.8428$ in (4)

$$x = \frac{1}{28}[32 - 4(1.5476) + 1.8428]$$

$$= 0.9876$$

Put $x = 0.9876$ and $z = 1.8428$ in (5)

$$y = \frac{1}{17}[35 - 2(0.9876) - 4(1.8428)]$$

$$= 1.5090$$

Put $x = 0.9876$ and $y = 1.5090$ in (6)

we get

$$z = \frac{1}{10}[24 - 0.9876 - 3(1.5090)]$$

$$= 1.8485$$

After third iteration, we have $x =$ 0.9876, $y = 1.5090$, $z = 1.8485$

Fourth iteration:

Put $y = 1.5090$, and $z = 1.8485$ in ④, we get

$$x = \frac{1}{28}[32 - 4y + z]$$

$$= \frac{1}{28}[32 - 4(1.5090) + 1.8485]$$

$$= 0.9933$$

Put $x = 0.9933$ and $z = 1.8485$ in ⑤ we get

$$y = \frac{1}{17}[35 - 2x - 4z]$$ 36

$$= \frac{1}{17}[35 - 2(0.9933) - 4(1.8485)]$$

$$= \frac{1}{17}[35 - 1.9866 - 7.394]$$

$$= 1.5070$$

Put $x = 0.9933$, $y = 1.5070$ in ⑥ we get

$$z = \frac{1}{10}[24 - x - 3y]$$

$$= \frac{1}{10}[24 - 0.9933 - 3 \times 1.5070]$$

$$= \frac{1}{10}[24 - 0.9933 - 4.521]$$

$$= 1.8486$$

After fourth iteration, we have

$$x = 0.9933,\ y = 1.5070,\ z = 1.8486$$

Fifth iteration:-

Put $y = 1.507$, $z = 1.8486$ in ④

$$x = \frac{1}{28}[32 - 6.028 + 1.8486]$$

$$= 0.9936$$

Put $x = 0.9936$, $z = 1.8486$ in ⑤

$$y = \frac{1}{17}[35 - 1.9872 - 7.3944]$$

$$= 1.5070$$

Put $x = 0.9936$, $y = 1.5070$ in ⑥

we get

$z = \frac{1}{10}[24 - x - 3y]$

$= \frac{1}{10}[24 - 0.9936 - 4.521]$

$= 1.8485$

After fifth iteration, we have

$x = 0.9936$, $y = 1.5070$, $z = 1.8485$

∴ Fourth and fifth iterations gives almost the same values

37

ii, Gauss Jacobi method.

Gauss iteration formula is

$$x^{r+1} = \frac{1}{a_1}[d_1 - b_1 y^{(r)} - c_1 z^{(r)}]$$

$$y^{r+1} = \frac{1}{b_2}[d_2 - a_2 x^{(r)} - c_2 z^{(r)}]$$

$$z^{r+1} = \frac{1}{c_3}[d_3 - a_3 x^{(r)} - b_3 y^{(r)}].$$

First Iteration.

Let the initial values be $x = y = z = 0$

Put $x = y = z = 0$ in ④ we get

$x_1 = \frac{32}{28} = 1.1429$ $\quad y_1 = \frac{35}{17} = 2.0588$

$z_1 = \frac{24}{10} = 2.4$

Second Iteration:-

$x_2 = \frac{1}{28}[32 - 4y_1 + z_1]$

$= \frac{1}{28}[32 - 8.2352 + 2.4] = 0.9345$

$y_2 = \frac{1}{17}[35 - 2x_1 - 4z_1] = \frac{1}{17}[35 - 2.2858 - 9.6] = 1.3597$

$z_2 = \frac{1}{10}[24 - x - 3y_1] = \frac{1}{10}[24 - 1.1429 - 6.1764]$

$= 1.6681.$

Third Iteration:

$x_3 = \frac{1}{28}[32 - 4y_2 + z_2]$

$= \frac{1}{28}[32 - 5.4388 + 1.6681]$

$= 1.0082$

$$y_3 = \frac{1}{17}[35 - 2x_2 - 4z_2] = \frac{1}{17}[35 - 1.869 - 6.6\ldots]$$
$$= 1.5564$$
$$z_3 = \frac{1}{10}[24 - x_2 - 3y_2] = \frac{1}{10}[24 - 0.9345 - 4.0791]$$
$$= 1.8986$$

Fourth iteration:

$$x_4 = \frac{1}{28}[32 - 4y_3 + z_3]$$
$$= \frac{1}{28}[32 - 6.2256 + 1.8986]$$
$$= 0.9883$$

38

$$y_4 = \frac{1}{17}[35 - 2x_3 - 4z_3]$$
$$= \frac{1}{17}[35 - 2.0164 - 7.5944]$$
$$= 1.4935$$
$$z_4 = \frac{1}{10}[24 - x_3 - 3y_3] = \frac{1}{10}[24 - 1.0082 - 4.6692]$$
$$= 1.8323$$

Fifth iteration:

$$x_5 = \frac{1}{28}[32 - 4y_4 + z_4]$$
$$= \frac{1}{28}[32 - 5.974 + 1.8323] = 0.9949$$
$$y_5 = \frac{1}{17}[35 - 2x_4 - 4z_4] = \frac{1}{17}[35 - 1.9766 - 7.3292]$$
$$= 1.5114$$
$$z_5 = \frac{1}{10}[24 - x_4 - 3y_4]$$
$$= \frac{1}{10}[24 - 0.9883 - 4.4805]$$
$$= 1.8531$$

Sixth Iteration:

$$x_6 = \frac{1}{28}[32 - 4y_5 + z_5]$$
$$= \frac{1}{28}[32 - 6.0456 + 1.8531]$$
$$= 0.9931$$
$$y_6 = \frac{1}{17}[35 - 2x_6 - 4z_6]$$
$$= \frac{1}{17}[35 - 1.9898 - 7.4124]$$
$$= 1.5058$$

$z_6 = \frac{1}{10}[24 - x_5 - 3y_5]$

$= \frac{1}{10}[24 - 0.9949 - 4.5342] = 1.8471$

Seventh iteration:

$x_7 = \frac{1}{28}[32 - 4y_6 + z_6] = \frac{1}{28}[32 - 6.0232 + 1.8471]$

$= 0.9937$

$y_7 = \frac{1}{17}[35 - 2x_6 - 4z_6] = \frac{1}{17}[35 - 1.9862 - 7.3884]$

$= 1.5074$

$z_7 = \frac{1}{10}[24 - x_6 - 3y_6] = \frac{1}{10}[24 - 0.9931 - 4.5174]$

$= 1.8490$ 39

Eighth iteration:

$x_8 = \frac{1}{28}[32 - 4y_7 + z_7] = \frac{1}{28}[32 - 6.0296 + 1.8490]$

$= 0.9936$

$y_8 = \frac{1}{17}[35 - 2x_7 - 4z_7] = \frac{1}{17}[35 - 1.9874 - 7.396]$

$= 1.5069$

$z_8 = \frac{1}{10}[24 - x_7 - 3y_7] = \frac{1}{10}[24 - 0.9937 - 4.5222]$

$= 1.8484$

The values in the seventh and Eighth iterations are close to each other.

Hence the solution is given by $x = 0.9936$

$y = 1.5069$, $z = 1.8484$

Remark:- From the above we ~~absolute~~ observe that the Gauss - seidel method gives the answer in Five iteration whereas Gauss Jacobi method gives is more or less than same answer only after eight iterations.

Unit 5

Unit – V

Numerical Solutions of Ordinary Differential Equations – Solution by Taylor Series, Picard's method of Successive approximations, Euler method, Modified Euler method Runge – Kutta Methods.

UNIT - V

Numerical solutions of ordinary Differential Equations

Taylor's series Method:

Consider the first Order differential Equ

$$\frac{dy}{dx} = f(x,y) \longrightarrow ①$$

with $y(x_0) = y_0$

Diff ① with respective to x, we get

$$\frac{d^2y}{dx^2} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y'$$

1

ie) $y'' = f_x + f_y y' \rightarrow ②$

Diff successively we can obtain $y'''; y''''; \ldots$

putting $x = x_0$ and $y = y_0$ we get $y_0', y_0'', y_0''', \ldots$

The Taylor's series expansion of $y(x)$ about $x = x_0$ is given by

$$y(x) = y(x_0) + (x-x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \cdots$$

$$= y_0 + (x-x_0) y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \cdots \rightarrow ③$$

Sub these values of $y_0', y_0'', y_0''', \ldots$ we obtain $y(x)$ for all values of x for which ③ convergent. Let $x_1 = x_0 + h$ and

Let

$$y(x_1) = y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} \cdot y_0'' + \cdots$$

once y_1 is known, we can compute $y_1', y_1'', \ldots$ from ①, ② etc

then y can be expand in a Taylor's Series about $x = x_1$, and we have

$$y(x_1+h) = y(x_2) = y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \cdots \rightarrow ④$$

Containing this way we find the solution $y(x)$

Problem:- 1 Using Taylor's method solve $\frac{dy}{dx} = 1 + xy$ with $y_0 = 2$, $x_0 = 0$. Find i) $y(0.1)$ (ii) $y(0.2)$ (iii) $y(0.3)$

i) The Taylor's algorithm is

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \dots \rightarrow ①$$

Here $x_0 = 0$, $y_0 = 2$ and $h = 0.1$

Given, $y' = \frac{dy}{dx} = 1 + xy \rightarrow ②$

Successively diff ② we get

$uv' = uv' + vu'$

$$y'' = \frac{d^2y}{dx^2} = y + xy'$$

$$y''' = 2y' + xy''$$

Now, $y_0' = y'_{(x_0, y_0)} = 1 + x_0 y_0 = 1$

$$y_0'' = (y'')_{(x_0, y_0)} = y_0 + x_0 y_0' = 2$$

$$y_0''' = (y''')(x_0, y_0) = 2y_0' + x_0 y_0'' = 2$$

using these in ① we get

$$y_1 = 2 + \frac{(0.1)}{1!} + \frac{(0.1)^2}{2!} \times 2 + \frac{(0.1)^3}{3!} \times 2$$

$$= 2.1103 \qquad \therefore y(0.1) = 2.1103$$

ii) The Taylor's algorithm for the next app is

$$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' \rightarrow ③$$

Now

$$y_1' = y'_{(x_1, y_1)} = 1 + x_1 y_1 = 1 + (0.1)(2.1103) = 1.21103$$

$$y_1'' = (y'')(x_1, y_1) = y_1 + x_1 y_1' = 2.1103 + (0.1)(1.21103) = 2.2314$$

$$y_1''' = (y''')(x_1, y_1) = 2y_1' + x_1 y_1'' = 2(1.21103) + (0.1)(2.2314) = 2.6452$$

$x_2 = x_1 + 0.1 = 0.2$

③ becomes

$$y_2 = 2.1103 + \frac{(0.1)}{1!}(1.21103) + \frac{(0.1)^2}{2!}(2.2314) + \frac{(0.1)^3}{3!}(2.6452)$$

$= 2.2430$

$\therefore y(0.2) = 2.2430$

iii) The Taylor's algorthim for third app is

$$y_3 = y_2 + \frac{h}{1!} y_2' + \frac{h^2}{2!} y_2'' + \frac{h^3}{3!} y_2''' \rightarrow (4)$$

Now

$1 + (0.2)(2.2430)$

$$y_2' = y'(x_2, y_2) = 1 + x_2 y_2 = 1.4486$$

$$y_2'' = y''(x_2, y_2) = y_2 + x_2 y_2' = 2.53272$$

$$y_2''' = y'''(x_2, y_2) = 2y_2' + x_2 y_2'' = 3.4037.$$

(4) becomes

$$y_3 = 2.2430 + (0.1)(1.4486) + \frac{(0.1)^2}{2!}(2.532\ 7\cdot) + \frac{(0.1)^3}{3!}(3.4037)$$

(3)

$= 2.4011$

$y(0.3) = 2.4011$

(2) using Tayloo's method find $y(0.1)$ correct to 3 decimal places from $\frac{dy}{dx} + 2xy = 1$, $y_0 = 0$

Solu:

Given $y' = \frac{dy}{dx} = 1 - 2xy$ ——— (1)

The Tayloo's Algooithm is

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots$$

Here $x_0 = 0$, $y_0 = 0$ and $h = 0.1$

diff (1) we get,

$y'' = -2(y + xy')$

$y''' = -2(xy'' + 2y')$

$\therefore y'(x_0, y_0) = y_0' = 1.$

$y''(x_0, y_0) = y_0'' = 0$

$y'''(x_0, y_0) = y_0''' = -4$

Sub the value of $y_0', y_0'', \ldots$

$$y_1 = 0 + 0.1 + \frac{(0.1)^2}{2!} \times 0 + \frac{(0.1)^3}{3!} \times (-4)$$

-0.004

-0.0006

$= 0.0993$

$\therefore y(0.1) = 0.0993$

3) using Tayloo's sebies method find y at $x = 1.1$ and 1.2 by solving $\frac{dy}{dx} = x^2 + y^2$ given $y(1) = 2.3$

Soln:

given $\frac{dy}{dx} = x^2+y^2$ —— ①

Here $x_0 = 1, y_0 = 2.3, h = 0.1$

i) The Taylor's series expansion is

$$y_1 = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots \quad \text{—— ②}$$

Diff ① w.r.to x we get

$y'' = 2x+2yy'$

$y''' = 2+2(yy''+y'^2)$

$\therefore y_0' = x_0^2+y_0^2 = 6.29$

$y_0'' = 2x_0+2y_0y_0' = 30.934$

$y_0''' = 2+2(y_0y_0''+y_0'^2) = 223.4246$

4

39.5

using this in ② we get,

$$y(1.1) = y_1 = 2.3 + \frac{0.1}{1}(6.29) + \frac{(0.1)^2}{2}(30.934) + \frac{(0.1)^3}{6}(223.4246)$$

$y_1 = 3.1209$

Here $x_1 = 1.1$ and $y_1 = 3.1209$

ii) we have the Taylor's expansion

$$y_2 = y_1 + \frac{h}{1!}y_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \cdots \quad \text{—— ③}$$

$y_1' = y'(x_1, y_1) = x_1^2+y_1^2 = 10.95$

$y_1'' = y''(x_1, y_1) = 2x_1+2y_1y' = 70.5477$

$y_1''' = y'''(x_1, y_1) = 2+2(y_1y_1''+y_1'^2) = 682.1496$

$\therefore$ ③ becomes,

$$y_2 = 3.1209 + 0.1(10.95) + \frac{(0.1)^2}{2}(70.5477) + \frac{(0.1)^3}{6}(682.1496)$$

$= 4.6823$

Hence $y(1.1) = 3.1209$ and $y(1.2) = 4.6823$

Picard's Method

consider the first order differential equation

$\frac{dy}{dx} = f(x,y)$ —— ①

with initial condition $y = y_0$ when $x = x_0$.

we now replace ① by an equivalent integral equation-ion. integrating ① between limits, we get,

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x,y)\,dx$$

$y(x_0) = $ $x_0 = 1, y_0 = 2.3$

$$\text{(i.e)}\; y = y_0 + \int_{x_0}^{x} f(x,y)\,dx \quad \text{—— ②}$$

This is an integral equation which contains the unknown y under the integral sign.

② is equivalent to ① since any solution of ② is a solution of ① and vice versa.

The first approximation y_1 to the solution is obtained by putting $y=y_0$ in $f(x,y)$ and from ② we have

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

Similiarly for the second approximation y_2, put $y=y_1$ in $f(x,y)$ and from ② we have

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx$$

Continuing this process the nth approximation is given by

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\,dx$$

This is known as picard's iteration formula.

Problem:

1, using picard's method solve $\frac{dy}{dx} = 1+xy$ with $y(0)=2$. find $y(0.1)$, $y(0.2)$ and $y(0.3)$

The picard's iteration for the differential equation $\frac{dy}{dx} = f(x,y)$ is $y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\,dx$

where $n = 1, 2, \cdots$

Given $f(x,y) = 1+xy$, $x_0 = 0$, $y_0 = 2$

$\therefore$ The first app is

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

$$= 2 + \int_{0}^{x} (1+2x)\,dx$$

$$= 2 + x + x^2$$

The second app is

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx$$

$$= 2 + \int_{0}^{x} [1 + x(2 + x + x^2)]\,dx$$

$$y_2 = 2 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

The third app is

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\,dx$$

$$= 2 + \int_{0}^{x} \left[1 + x\left(2 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{4}\right)\right] dx$$

$$= 2 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{15} + \frac{x^6}{24}$$

Putting $x = 0.1, 0.2, 0.3$ in y_1 y_2 and y_3

we get

$$y_1 = y(0.1) = 2 + 0.1 + (0.1)^2 = 2.11$$

$$y_2 = y(0.2) = 2 + (0.2) + (0.2)^2 + \frac{(0.2)^3}{3} + \frac{(0.2)^4}{4}$$
$$= 2.2431$$

$$y_3 = y(0.3) = 2 + (0.3) + (0.3)^2 + \frac{(0.3)^3}{3} + \frac{(0.3)^4}{4} + \frac{(0.3)^5}{15} + \frac{(0.3)^6}{24}$$

0.09 $0.009 + \frac{0.0081}{4} + = 0.002025$

$$= 2.4012$$

6

$\therefore$ $y(0.1) = y_1 = 2.11$

$y(0.2) = y_2 = 2.2431$

$y(0.3) = y_3 = 2.4012$.

2) Find the value of $y(0.1)$ by picard's method given $\frac{dy}{dx} = \frac{y-x}{y+x}$ $y(0) = 1$

Soln: The picard's iterative formula for the differential equation $\frac{dy}{dx} = f(x,y)$ is

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})dx \quad \text{where } n = 1, 2, \ldots$$

Here $f(x,y) = \frac{y-x}{y+x}$, $x_0 = 0$ & $y_0 = 1$

$\therefore$ The first app is

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)dx$$

$$= 1 + \int_0^x \left(\frac{1-x}{1+x}\right)dx$$

$$= 1 + \int_0^x \left(-1 + \frac{2}{1+x}\right)dx \quad \text{(By Pratial fraction)}$$

$$= 1 + \left[-x + 2\log_e(1+x)\right]_0^x$$

$$y_1 = 1 - x + 2\log_e(1+x)$$

putting $x = 0.1$, we get

$$y(0.1) = y_1 = 1 - 0.1 + 2\log_e(1+0.1)$$
$$= 0.9 + 2 \times 0.0953$$
$$= 1.0906$$
$$y(0.1) = 1.0906.$$

3) Find the successive approximate solution of the differential equation $y' = y$, $y(0) = 1$ by picard's method and compare it with the exact solution.

Picard's iteration formula is given by

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\,dx \quad \text{where } n = 1, 2, \ldots$$

Hence $f(x, y) = y$, $x_0 = 0$ and $y_0 = 1$

$\therefore$ The first app is

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

$$= 1 + \int_0^x dx$$

$$y_1 = 1 + x$$

The second app is

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx$$

$$= 1 + \int_0^x y_1\,dx$$

$$= 1 + \int_0^x (1+x)\,dx$$

$$y_2 = 1 + x + \frac{x^2}{2}$$

The third app is

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\,dx$$

$$= 1 + \int_0^x y_2\,dx$$

$$= 1 + \int_0^x \left(1 + x + \frac{x^2}{2}\right)dx$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

Proceeding like this we can find successive approximation

$$y' = y$$

(i.e) $\frac{dy}{dx} = y$

$\therefore \frac{dy}{y} = dx$

Integrating we get

$$\log_e y = x + c_1$$

$\therefore$ The exact solution is

$$y = e^{x + c_1} = c e^{x}$$

using the initial condition $x=0, y=1$

we get, $C=1$

(i.e) $y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

Hence the successive approximate solution are the partial of the exact solution

Problm: 4

Find an approximation solution of the initial value problem. $y' = 1+y^2$, $y(0)=0$, by picard's method and compare with the exact solution

soln:

Picard's iteration formula is

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\,dx \quad \text{Where } n=1,2,\cdots$$

Here, $f(x,y) = 1+y^2$, $x_0=0$ and $y_0=0$

8

$\therefore$ The first app is

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

$$y_1 = y_0 + \int_0^x (1+y_0^2)\,dx$$

$$= \int_0^x dx$$

$$y_1 = x$$

The second app is

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx$$

$$= y_0 + \int_0^x (1+y_1^2)\,dx$$

$$= y_0 + \int_0^x (1+x^2)\,dx$$

$$y_2 = x + \frac{x^3}{3}$$

The third app is

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\,dx$$

$$= y_0 + \int_0^x (1+y_2^2)\,dx$$

$$= y_0 + \int_0^x 1 + (x + \frac{x^3}{3})\,dx$$

$$= y_0 + \int_0^x (1+x^2+\frac{2x^4}{3}+\frac{x^6}{9})\,dx$$

$$y_3 = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{x^7}{63}$$

Proceeding like this we can find the further approximation solution.

Now, the diff equation is

$$\frac{dy}{dx} = 1+y^2$$

9

(i.e) $\frac{dy}{1+y^2} = dx$

Integrating, we get

$$\tan^{-1} y = x + c$$

Using the initial condition $x=0$ and $y=0$

we get, $C = 0$

$$\therefore y = \tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots$$

Clearly, the first three terms of y_3 are same as that of the exact solutions

Eulcer's method.

Consider the differential equation

$$\frac{dy}{dx} = f(x,y) \quad \text{with } y(x_0) = y_0 \longrightarrow (1)$$

Suppose we want to solve (1) for y at the points $x_r = x_0 + rh, r = 1, 2, 3, \ldots$

Integrating (1) between the limits x_0 and x_1

we get,

$$\int_{y_0}^{y_1} dy = \int_{x_0}^{x_1} f(x,y)\,dx$$

Hence

$$y_1 = y_0 + \int_{x_0}^{x_1} f(x,y)\,dx \longrightarrow (2)$$

Assuming that $f(x,y) = f(x_0, y_0)$ in $x_0 \le x \le x_1$

we get,

$$y_1 = y_0 + (x_1 - x_0) f(x_0, y_0)$$

$$\therefore y_1 = y_0 + h f(x_0, y_0) \longrightarrow (3)$$

III'ly if $x_1 \le x \le x_2$ we have

$$y_2 = y_1 + \int_{x_1}^{x_2} f(x,y)\,dx$$

sub $f(x_1, y_1)$ for $f(x,y)$, we get

$y_2 = y_1 + h f(x_1, y_1) \longrightarrow (4)$

proceeding like this we obtain the general form

$$y_{n+1} = y_n + h f(x_n, y_n), \quad n = 0, 1, 2, \ldots$$

This is called Eulcer's algorithm

since $x_n = x_0 + nh$ and

$y_n = y(x_n)$ the above formula can be also

written as $y(x+h) = y(x) + h f(x, y)$

(10)

Modified Eulcer's method

Instead of approximating $f(x,y)$ by $f(x_0, y_0)$ in (2) we applyit by $\frac{1}{2}\left[f(x_0,y_0) + f(x_1,y_1)\right]$ which is the mean of slopes of tangent at the points corresponding to $x = x_0$ and $x = x_1$ Thus we obtain

$$y_1^{(1)} = y_0 + \left[f(x_0, y_0) + f(x_1, y_1)\right]$$

Where y_1 is given by (2) . $y_1^{(1)}$ is the first modified value of y_1

$$\text{Let } y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(1)})\right]$$

we repeat this process till two consecutive values of y agree . Let y_1 be the final value obtained to the desired accuracy using this value of y_1 we compute

$$y_2 = y_1 + h f(x_0 + h, y_1)$$

Now, Let $y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_0+h, y_1) + f(x_0+2h, y_2)\right]$

we repeat this process until two consecutive value agree . then we proceed to calculate y_3 as above and continue the process till we calculate y_n .

Prblm:- 1 Solve $\frac{dy}{dx} = 1 - y$, $y(0) = 0$ using Eulcer's method find y at $x = 0.1$ and $x = 0.2$ compute the result with results of the exact solution.

soln:- The Eulcer's formula for the numerical solution of the diff equation $\frac{dy}{dx} = f(x,y)$ is

$$y_{n+1} = y_n + h f(x_n, y_n) \rightarrow ①$$

The given diff equation is $\frac{dy}{dx} = 1-y$

$\therefore f(x,y) = 1-y$

Also, we have $x_0 = 0$, $y_0 = 0$, $h = 0.1$

Put $n=0$ in ① we get

$y(0.1) = y_1 = y_0 + h f(x_0, y_0)$

$= 0 + (0.1)(1) = 0.1$

Now, $x_1 = x_0 + h = 0 + 0.1 = 0.1$.

Put $n=1$ in ① we get $y(0.2) = y_2 = y_1 + f(x_1, y_1)$

$= 0.1 + 0.1(1-0.1)$

$= 0.19$

Hence, $y(0.1) = 0.1$ and $y(0.2) = 0.19$

The exact solution of $\frac{dy}{dx} = 1-y$ is got from

$\frac{dy}{1-y} = dx \quad \therefore \log(1-y) = x + c$

Put $x=0$ & $y=0$ we get $c=0$

$\therefore 1-y = e^x$. Hence $y = 1-e^x$

$y(0.1) = 1 - e^{0.1} = 0.1052$ and $y(0.2) = 1 - e^{0.2}$

$= 0.2214$.

2) using Euler's method solve $\frac{dy}{dx} = 1+xy$ with $y(0)=2$ find $y(0.1)$, $y(0.2)$ & $y(0.3)$. Also find the values by modified Euler's method.

soln: The Euler's formula for numerical soln of the diff. equation $\frac{dy}{dx} = f(x,y)$ is

$$y_{n+1} = y_n + h f(x_n, y_n) \quad n = 0,1,2,\dots \rightarrow ①$$

Here $f(x,y) = 1+xy$, $x_0 = 0$, $y_0 = 2$ and $h = 0.1$

Put $n=0$ in ① we get $y(0.1) = y_1 = y_0 + hf(x_0, y_0)$

$= 2 + (0.1) f(0, 2)$

$= 2.1$

Now, $x_1 = x_0 + h = 0.1$

put $n=1$ ① we get $y(0.2) = y_2 = y_1 + hf(x_1, y_1)$

$= 2.1 + (0.1)[1 + 0.1 \times 2.1]$

$= 2.221$

Now $x_2 = x_0 + 2h = 0.2$

put $n=2$ in ① we get $y(0.3) = y_3 = y_2 + hf(x_2, y_2)$

$= 2.221 + (0.1)[1 + 0.2 \times 2.221]$

12

$= 2.3654$

Hence $y(0.1) = 2.1$, $y(0.2) = 2.221$, $y(0.3) = 2.3654$

Modified Euler's Method.

Starting value for $y = 2.1$

$y_1^{(1)} = y_0 + h/2 [f(x_0, y_0) + f(x_1, y_1)]$ $\frac{1}{x+y}$

$= 2 + \frac{0.1}{2}[1 + 1 + (0.1)(2.1)]$ $\log(x+y)$

$= 2.2205$

$y_1^{(2)} = y_0 + h/2 [f(x_0, y_0) + f(x_1, y_1^{(1)})]$

$= 2 + \frac{0.1}{2}[1 + 1 + (0.1)(2.2205)] = 2.1111$

Continuing this process we get $y_1^{(3)} = 2.1105$

$y_1^{(4)} = 2.1105$

$\therefore$ Find value of $y_1 = 2.1105$

Now, starting value of

$y_2 = y_1 + h f(x_0 + h, y_1)$

$= 2.1105 + 0.1[1 + (0.1)(2.1105)]$

$= 2.2316$

$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_0 + h, y_1) + f(x_0 + 2h, y_2)]$

$= 2.1105 + \frac{0.1}{2}[1 + (0.1)(2.1105) + 1 + (0.2)(2.2316)]$

$= 2.2434$

Continuing this process we get

$y_2^{(2)} = 2.2434$, $y_2^{(3)} = 2.2434$

$\therefore$ Find value of $y_2 = 2.2434$

3) starting value of $y_3 = y_2 + h f(x_0 + 2h, y_2)$

$= 2.2434 + (0.1)[1 + (0.2)(2.2434)]$

$= 2.2579$

~~Continuing~~ $y_3^{(1)} = y_2 + h/2 [f(x_0 + 2h, y_2) + f[x_0 + 3h, y_3]]$

$= 2.3997$

continuing this process we get $y_3^{(2)} = 2.4018$ $y_3^{(3)} = 2.4019$, $y_3^{(4)} = 2.4019$

$\therefore$ Final value of $y_3 = 2.4019$

Hence $y_1 = 2.1105$, $y_2 = 2.3997$, & $y_3 = 2.4019$

3. Given $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$, $y(1) = 1$ evaluate $y(1.3)$ by modified Euler's method.

The given equation is

$$\frac{dy}{dx} = \frac{1}{x^2} - \frac{y}{x} = \frac{1-xy}{x^2}, \quad y(1) = 1$$

$\therefore f(x,y) = \frac{1-xy}{x^2}$, $x_0 = 1$, $y_0 = 1$ and we take $h = 0.1$. Starting value of $y(1.1) = y_1$ is given by.

$$y_1 = y_0 + h f(x_0, y_0) = 1$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x, y_1)\right]$$

$$= 1 + \frac{0.1}{2}\left[0 + f(1.1, 1)\right]$$

$$= 1 + 0.05\left[\frac{1 - 1.1}{(1.1)^2}\right] = 0.9959$$

$$y_1^{(2)} = 1 + \frac{h}{2}\left[f(x_0, y_0) + f(x, y_1^{(1)})\right]$$

$$= 1 + 0.05\left[\frac{0 + 1 - (1.1) \times (0.9959)}{(1.1)^2}\right]$$

$$= 0.9605$$

continuing this process we get

$y_1^{(3)} = 0.9977$ $y_1^{(4)} = 0.9960$, $y_2^{(5)} = 0.9960$.

$\therefore$ Final value of $y_1 = 0.9960$

Starting value of $y_2 = y_1 + h f(x_0 + h, y_1)$

$$= 0.9960 + 0.1\left[\frac{1 - (1.1)(0.9960)}{(1.1)^2}\right]$$

$$= 0.9881.$$

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_0 + h, y_1) + f(x_0 + 2h, y_2)\right]$$

$= 0.9960 + (0.05)[f(1.1, 0.9960)] + [f(1.2, 0.9881)]$

$= 0.9856$

Continuting this process we get $y_2^{(2)} = 0.9857$, $y_2^{(3)} = 0.9857$ $\therefore$ Final value of $y_2 = 0.9857$

Starting Value of $y_3 = y_2 + 0.1[f(1.2, 0.9857)]$

$= 0.9730$

Now, $y_3^{(1)} = y_2 + \frac{h}{2}[f(x_0+2h, y_2) + f(x_0+3h, y_0)]$

14

$= 0.9857 + 0.05[f(1.2, 0.9858) + f(1.3, 0.930)]$

$= 0.971$

cotinuting this process we get $y_3^{(2)} = 0.9716$, $y_3^{(3)} = 0.9662$ $y_3^{(4)} = 0.9662$ $\therefore y(1.3) =$ final value of $y_3 = 0.9662$.

Runge kutta Method:

The Runge - kutta methods do not require the calculations of higher order derivatives and they are designed to give greates accuracy with the advantage of requiring only the function values at some selected points on the sub-interval these methods agree with taylor's series solution upto terms of h^r. where r is the Order of the Runge-kutta method [R.k method]

First order R.K method:

Consider $\frac{dy}{dx} = f(x,y)$ with $y(x_0) = y_0 \rightarrow$ ①

The Euler's Formula for first app. to the solution of the above differential equation is given by.

$y_1 = y_0 + h\, f(x_0, y_0)$

$y_1 = y_0 + h\, y_0'$ $\quad [\because y' = f(x,y)]$

Also $y_1 = y(x_0+h) = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \cdots$

Clearly, the Euler's method agree with the Taylor's series solution upto the term in h. Hence Euler's method is the Runge-kutta method of first order

15

II. Second order R.K method:

The modified Euler's Formula for (1) is

$$y_1 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_0 + h_0, y_0 + hf(x_0, y_0))\right]$$

ie) $y_1 = y_0 + \frac{h}{2}\left[f_0 + f(x_0 + h, y_0 + hf_0)\right] \rightarrow (2)$

where $f_0 = f(x_0, y_0)$

Expanding the L.H.S by Taylor's series we get

$$y_1 = y(x_0 + h) = y_0 + \frac{h}{1!}y'_0 + \frac{h^2}{2!}y''_0 + \frac{h^3}{3!}y'''_0 \ldots \rightarrow (3)$$

Expanding $f(x_0 + h, y_0 + hf_0)$ by taylor's series for the function of two variables we have

$$f(x_0 + h, y_0 + hf_0) = f(x_0, y_0) + \frac{h}{1!}\left[\left(\frac{\partial f}{\partial x}\right)_{(x_0, y_0)} + f_0\left(\frac{\partial f}{\partial y}\right)(x_0, y_0) + O(h^2)\right]$$

using this in (2) we get

$$y_1 = y_0 + \frac{h}{2}\left[f_0 + \left\{f(x_0, y_0) + h\left(\frac{\partial f}{\partial x}\right)_{(x_0, y_0)} + hf_0\left(\frac{\partial f}{\partial y}\right)_{(x_0, y_0)} + O(h^2)\right]\right.$$

$$= y_0 + \frac{1}{2}\left[hf_0 + hf_0 + h^2\left\{\left(\frac{\partial f}{\partial x}\right)_{(x_0, y_0)} + f_0\left(\frac{\partial f}{\partial y}\right)_{(x_0, y_0)} + O(h^3)\right]\right.$$

$$= y_0 + hf_0 + \frac{h^2}{2!}f'_0 + O(h^3)$$

$$x = y_0 + hy'_0 + \frac{h^2}{2!}y''_0 + O(h^3) \rightarrow (4)$$

Comparing (3) & (4) we find that the modified Euler's method agrees with the Taylor's series solution upto the h^2 term

Hence the modified Euler's method is the R.k method of 2nd order

The second order of R·k formula is

$$y_1 = y_0 + \frac{1}{2}(k_1+k_2)$$

where $k_1 = hf(x_0, y_0)$ and $k_2 = hf(x_0+h, y_0+k_1)$

Third order R·k method:

Third Order R·k method is given by

$$y_1 = y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3)$$

Where $k_1 = hf(x_0, y_0)$

16

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$

$k_3 = h(x_0+h, y_0+k')$

Where $k' = hf(x_0+h, y_0+k_2)$

Fourth Order R·k method:

This method is most commonly used and is referred as the R·k method

The working rule for solving problem

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0$$

By Fourth Order R·k method is as follows Calculate succesively

$$k_1 = hf(x_0, y_0)$$
$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$
$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$
$$k_4 = hf(x_0+h, y_0+k_3) \text{ and}$$
$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Then the required app value is given by

$$y_1 = y_0 + \Delta y$$

Similarly, the value of y in the 2nd interval is obtained by replacing x_0 by x_1 and y_0 by y_1 in the above set of formulae and we obtain y_2. In general to find y_n sub x_{n-1} y_{n-1} in the expression for k_1, k_2 etc

Note: 1

The operation is identical whether the diff equation is linear or non-linear

$k_1 = hf(x_1, y_1)$

$= 0.1\ [0.1 \times 1.1169 + (1.1169)^2]$

$= 0.1359$

$k_2 = 0.1\ [0.05\ (1.05) + (1.05)^2]$

$= 0.1155$

$k_3 = hf[x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}]$

$= (0.1)[0.15\ (1.196) + (1.196)^2]$

$= 0.1610$

$k_4 = hf\ (x_1 + h, y_1 + k_3)$

$= 0.1\ [(0.2)(1.2779) + (1.2779)^2]$

$= 0.1889$

18

$\Delta y = \frac{1}{6}\ [k_1 + 2k_2 + 2k_3 + k_4]$

$= \frac{1}{6}[\ 0.1359 + 0.3164 + 0.3220 + 0.1889]$

$\Delta y = 0.1605$

$\therefore y_2 = y_1 + \Delta y$

$= 1.1169 + 0.1605$

$= 1.2774$

$\therefore y(0.2) = 1.2774.$

2) Use Runge-Kutta method of the fourth order to find $y(0.1)$ given that $\frac{dy}{dx} = \frac{1}{x+y}$, $y(0) = 1$

Soln: The formula fourth order R.k method is given by

$k_1 = hf\ (x_0, y_0)$

$k_2 = hf\ (x_0 + h/2,\ y_0 + k_1/2)$

$k_3 = hf\ (x_0 + h/2,\ y_0 + k_2/2)$

$k_4 = hf\ (x_0 + h,\ y_0 + k_3)$

$\Delta y = \frac{1}{6}\ (k_1 + 2k_2 + 2k_3 + k_4)$

where h is the interval of differencing and (x_0, y_0) is the enital value

Here $f(x, y) = \frac{1}{x+y}$, $x_0 = 0$, $y_0 = 1$

and $h = 0.1$

Now, $k_1 = (0.1)\ (\frac{1}{0+1}) = 0.1$

$$k_2 = (0.1)\left[\frac{1}{(x_0+h/2)(y_0+k_1/2)}\right]$$

$$= \frac{0.1}{0.5+1.05} = (0.1)\left[\frac{1}{1.1}\right]$$

$$= 0.0909$$

$$k_3 = 0.1\left[\frac{-1}{(x_0+h/2)(y_0+k_2/2)}\right]$$

$$= \frac{0.1}{0.5+0.45} = 0.0913$$

$$k_4 = (0.1)\left[\frac{1}{(x_0+h)+(y_0+k_3)}\right]$$

19

$$= \frac{0.1}{0.1+1.0913} = 0.0837$$

$$\Delta y = \frac{1}{6}\left[0.H\ 2(0.0909)+2(0.0913)+0.0839\right]$$

$$= 0.0914$$

$$y_1 = y_0 + \Delta y = 1+0.0914$$
$$= 1.0914$$

$$y(0.1) = 1.0914$$

Problm:-3
Using Fourth Order Runge-kutta method Evaluate the value of y when $x=1.1$ given that

$$\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2} \quad y(1)=1$$

Soln:- The formula for the fourth order R.K method of the differential equation

$$\frac{dy}{dx} = f(x,y) \text{ is given by}$$

$$k_1 = h\,f(x_0, y_0)$$

$$k_2 = hf(x_0+h/2,\ y_0+k_1/2)$$

$$k_3 = hf(x_0+h/2,\ y_0+k_2/2)$$

$$k_4 = hf(x_0+h,\ y_0+k_3)$$

$$\Delta y = \frac{1}{6}(k_1+2k_2+2k_3+k_4)$$

where h is the interval of differencing and (x_0, y_0) is the inital value

Here $f(x, y) = \frac{1}{x^2} - y/x$ $x_0 = 1$ and $y_0 = 1$ and $y_0 = 1$, $h = 0.1$

Now, $k_1 = (0.1)\left(\frac{1}{1^2} - \frac{1}{1}\right) = 0$

$$k_2 = (0.1)\left(\frac{1}{(x_0+h/2)^2} - \frac{y_0 + k_1/2}{x_0 + h/2}\right)$$

$$= (0.1)\left(\frac{1}{(1+\frac{0.1}{2})^2} - \frac{1+0}{1+0.1/2}\right)$$

20

$$= (0.1)(0.9070 - 0.9524)$$

$$= -0.00454$$

$$k_3 = (0.1)\left(0.9070 - \frac{1 + \left(\frac{-0.00454}{2}\right)}{1.05}\right)$$

$$= (0.1)(0.9070 - 0.9502)$$

$$= -0.00432$$

$$k_4 = (0.1)\left(\frac{1}{(1.1)^2} - \frac{1 - 0.00432}{1.1}\right)$$

$$= (0.1)(0.8264 - 0.9052)$$

$$= -0.00788$$

$$\therefore \Delta y = \frac{1}{6}(0 - 0.00908 - 0.00864 - 0.00788)$$

$$= -0.0042667$$

$$\therefore y_1 = y(1.1) = y_0 + \Delta y = 1 + (-0.0042667)$$

$$= 0.9957$$

Predictor corrector Method:

consider the equation $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$ we divide the range for x into a number of step sizes of equal width h. If x_i and x_{i+1} are two consectutive points then $x_{i+1} = x_i + h$ Eulcer's formula for the above differential equation is $y_{i+1} = y_i + h f(x_i, y_i)$ $\quad i = 1, 2, 3 \longrightarrow (1)$

The modified Eucler's formula

$$y_{i+1} = y_i + h/2 \left[f(x_i, y_i) + f(x_{i+1}, y_{i+1})\right]$$

$i = 1, 2, 3 \longrightarrow (2)$

The value of y_{i+1} is first eliminated by (1) and this value is inserted on the right side of (2) to get a better app for y_{i+1}. this value of y_{i+1} is again sub in (2) to find a still better app of y_{i+1} this Process is repeated until two consecutive value of y_{i+1} are almost equal

This technique of refining an initially crude estimate by means of more accurate formula is known as Predictor - corrector method

Equation (1) is called the Predictor and (2) is called corrector

21

In the method described so far, to solve a differential equation over an interval only the value of y at the begining of the interval was required. In the predictor corrector methods four prior value of y are needed to evaluate the value of y at x_{i+1}

A predictor formula is used to predictor the value y_{i+1} of y at x_{i+1} and then a corrector formula is used to improve the value of y_{i+1}

Milne's Method:

Consider the first order differential equation

$$dy/dx = f(x,y). \text{ with } y(x_0)=y_0$$

Newton's forward difference formula can be written as

$$f(x,y) = f_0 + n\Delta f_0 + \frac{n(n-1)}{2!}\Delta^2 f_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 f_0 + \cdots \longrightarrow (1)$$

Substituting this in the relation

$$y_4 = y_0 + \int_{x_0}^{x_0+4h} f(x,y)dx$$

We get $$y_4 = y_0 + \int_{x_0}^{x_0+4h}\left[f_0 + n\Delta f_0 + \frac{n(n-1)}{2!}\Delta^2 f_0 + \cdots\right]dx$$

put $x = x_0 + nh$ Hence $dx = h\,dn$

when $x = x_0$, $n = 0$ and when $x = x_0 + 4h$, $n = 4$

$$\therefore y_4 = y_0 + h \int_0^4 \left[f_0 + n\Delta f_0 + \frac{n(n-1)}{2!} \Delta^2 f_0 + \cdots \right] dn$$

$$= y_0 + h \left[4f_0 + 8\Delta f_0 + \frac{20}{3} \Delta^2 f_0 + \frac{8}{3} \Delta^3 f_0 + \cdots \right]$$

$$= y_0 + h \left[4y'_0 + 8(E-1)y'_0 + \frac{20}{3}(E^2 - 2E - 1)y'_0 + \frac{8}{3}(E^3 - 3E^2 - 3E - 1)y'_0 \right]$$

22

(neglacting fourth and higher order differences)

$$= y_0 + h \left[4y'_0 + 8(y'_1 - y'_0) + \frac{20}{3}(y'_2 - 2y'_1 + y'_0) + \frac{8}{3}(y'_3 - 3y'_2 + 3y'_1 - y'_0) \right]$$

$$= y_0 + h \left[\frac{2}{3} y'_1 - \frac{4}{3} y'_2 + \frac{8}{3} y'_3 \right]$$

Thus $y_4 = y_0 + \frac{4h}{3} \left[2y'_1 - y'_2 + 2y'_3 \right]$

since $x_0, x_1, \ldots x_n$ are any five consecutive values of x. the above equations can be written as

$$y_{n+1,P} = y_{n-3} + \frac{4h}{3} \left[y'_{n-2} - y'_{n-1} + 2y'_n \right] \rightarrow ②$$

This is called Milne's predictor method formula (the subscript p indicator that it is a predicted value)

This formula can be used to predict the value of y_4 when those $y_0, y_1, y_2, y_3, \ldots$ are known. To get a corrector formula we substitute Newton's formula ① in the relation

$$y_2 = y_0 + \int_{x_0}^{x_0+2h} f(x,y)\,dx$$

we get $y_2 = y_0 + \int_{x_0}^{x_0+2h} \left[f_0 + n\Delta f_0 + \frac{n(n-1)}{2!} \Delta^2 f_0 + \cdots \right] dx$

$$= y_0 + h \int_0^2 \left[f_0 + n\Delta f_0 + \frac{n(n-1)}{2!} \Delta^2 f_0 + \cdots \right]$$

[putting $x = x_0 + h$]

$= y_0 + h\left[2f_0 + 2\Delta f_0 + \frac{1}{3}\Delta^2 f_0 + \dots\right]$

$= y_0 + h\left[2y'_0 + 2(E-1)y'_0 + \frac{1}{3}(E^2 - 2E + 1)y'_0\right]$

neglating higher order diff

$= y_0 + h\left[2y'_0 + 2(y'_1 - y'_0) + \frac{1}{3}(y'_2 - 2y'_1 + y'_0)\right]$

Thus $y_2 = y_0 + \frac{h}{3}\left[y'_0 + 4y'_1 + y'_2\right]$ 23

since x_0, x_1, x_2 are any three consecutive value of x. the above relation can be written as

$$y_{n+1, c} = y_{n-1} + \frac{h}{3}\left[y'_{n-1} + 4y'_n - y_{n-1}\right] \longrightarrow (2)$$

This is known as Milne's corrector formula where the suffix c stands for corrector

An improved value of y_{n+1} is computed and again the corrector formula is applied until we get y_{n+1} to the desired accuracy

1. Given $dy/dx = 1/x+y$; $y(0) = 2$ If $y(0.2) = 2.09$, $y(0.4) = 2.17$ and $y(0.6) = 2.24$ find $y(0.8)$ using milne's method.

Milne's predictor formula is

$$y_{n+1, P} = y_{n-3} + \frac{4h}{3}\left[2y'_{n-2} - y'_{n-1} + 2y'_n\right] \rightarrow (1)$$

put $n = 3$ in (1) we get

$$y_{4, P} = y_0 + \frac{4h}{3}\left[2y'_1 - y'_2 + 2y'_3\right] \rightarrow (2)$$

we are given that $y_0 = x$, $y_1 = 2.09$

$y_2 = 2.17$ $\quad y_3 = 2.24$ and $h = 0.2$

The given diff equ is

$$y' = \frac{1}{x+y} \rightarrow (3)$$

From the above equation we calculate y'_1, y'_2, y'_3

$$y'_1 = \left(\frac{1}{x+y}\right)_{(x_1, y_1)} = \frac{1}{0.2 + 2.09} = 0.4367$$

$$y'_2 = \left(\frac{1}{x+y}\right)_{(x_2, y_2)} = \frac{1}{0.4 + 2.17} = 0.3891$$

$y'_3 = \left(\frac{1}{x+y}\right)(x_3, y_3) = \frac{1}{0.6+2.24} = 0.3521$

sub these values in ② we get

$y_4, P = 2 + \frac{4\times0.2}{3}(2\times0.4367 - 0.3891 + 2\times0.3521)$

$= 2.3169$ (correct to 4 decimal places)

Milne's corrector formula is $\to$ ④

$y_{n+1}, c = y_{n-1} + \frac{h}{3}(y'_{n-1} + 4y'_n + y'_{n+1})$ $\to$ ⑤

24

putting $n=3$ in ⑤ we get

$y_4, c = y_2 + \frac{h}{3}(y'_2 + 4y'_3 + y'_4) \to$ ⑥

Now, $y'_4 = \left(\frac{1}{x+y}\right)(x_4, y_4) = \frac{1}{0.8+2.3169}$ (using 4)

∴ ⑥ becomes $= 0.3208$

$y_4, c = 2.17 + \frac{0.2}{3}(0.3891 + 4\times0.3521 + 0.3208)$

$= 2.3112$ (correct to 4 decimal places)

Hence $y(0.8) = 2.3112$

2. using Milne's predictor corrector method find $y(0.4)$ for the diff-equ $dy/dx = 1+xy$, $y(0) = 2$

Milne's predictor formula is

$y_4, P = y_0 + \frac{4h}{3}[2y'_1 - y'_2 + 2y'_3] \to$ ①

Hence, $x_0 = 0$, $y_0 = 2$, $h = 0.1$

By Taylor's series method we have

$x_1 = 0.1$ $\quad y_1 = 2.1103$

$x_2 = 0.2$ $\quad y_2 = 2.2430$

$x_3 = 0.3$ $\quad y_3 = 2.4011$

Now, $y'_1 = (y')(x_1, y_1) = 1 + (0.1)(2.1103) = 1.21103$

$y'_2 = (y')(x_2, y_2) = 1 + (0.2)(2.243) = 1.4486$

$y'_3 = (y')(x_3, y_3) = 1 + (0.3)(2.4011) = 1.72033$

put the values in ① we get

$y_4, P = \frac{2 + 4(0.1)}{3}[2(1.21103) - 1.4486 + 2(1.72033)]$

$= 2.5885$

Milne's corrector formula is $y_{4,c} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4') \rightarrow ②$

Now,

$y_4' = (y')(x_4, y_4) = 1 + (0.4)(2.5885)$
$= 2.0354$

$\therefore$ ② becomes

$y_{4,c} = 2.243 + \frac{0.1}{3}[1.4486 + 4(1.72033) + 4(2.0354)]$
$= 2.5885$

Hence $y(0.4) = 2.5885$ 25

3. Given $\frac{dy}{dx} = \frac{1}{2}(1+x^2)y^2$ and $y(0) = 1$, $y(0.1) = 1.06$, $y(0.2) = 1.12$; $y(0.3) = 1.21$. Evaluate $y(0.4)$ by Milne's Predictor corrector method

Milne's predictor formula is

$y_{n+1,P} = y_{n-3} + \frac{4h}{3}[2y'_{n-2} - y'_{n-1} + 2y'_n]$

Put $n = 3$, we get $y_{4,P} = y_0 + \frac{4h}{3}[2y_1' - y_2' + 2y_3'] \rightarrow ①$

Here $x_0 = 0$, $y_0 = 1$, $x_1 = 0.1$, $y_1 = 1.06$
$x_2 = 0.2$, $y_2 = 1.12$, $x_3 = 0.3$ $y_3 = 1.21$

Now, $y_1' = (y')(x_1, y_1) = \frac{1}{2}(1 + x_1^2)y_1^2$
$= \frac{1}{2}(1 + (0.1)^2)(1.06)^2$
$= 0.5674$

$y_2' = (y')(x_2, y_2) = \frac{1}{2}(1 + x_2^2)y_2^2$
$= \frac{1}{2}(1 + (0.2)^2)(1.14)^2$
$= 0.5674$

$y_3' = (y')(x_3, y_3) = \frac{1}{2}(1 + x_3^2)y_3^2$
$= \frac{1}{3}(1 + 0.3^2)(1.12)^2$
$= 0.7979$

put these values in ① we get

$y_{4,P} = \frac{1 + 4(0.1)}{3}[2(0.5674) - 0.6523 + 2(0.7979)]$

$y_{n+1,P} = y_{n-3} + \frac{4h}{3}[2y'_{n-2} - y'_{n-1} + 2y'_n]$

$= 1.2771$

Milne's corrector formula

$y_{n+1,c} = y_{n-1} + \frac{h}{3}[y'_{n-1} + 4y'_n - y'_{n+1}]$

Put $n=3$ we get

$y_{4,c} = y_2 + \frac{h}{3}(y'_2 + 4y'_3 + y'_4) \rightarrow ②$

Now, $y'_4 = (y')(x_4, y_4) = \frac{1}{2}(1+x_4^2)y_4^2$

$= \frac{1}{2}(1+(0.4)^2)(1.2771)^2$

$= 0.9460$

26

$\therefore$ ② becomes, $y_{4,c} = 1.12 + \frac{0.1}{3}(0.6523 + 4(0.7979) + 0.9460)$

$\therefore y(0.4) = 1.2797$

Now, $y'_4 = (y')(x_4, y_4) = \frac{1}{2}(1+x_4^2)y_4^2$

$= \frac{1}{2}(1+0.4^2)(1.2797)^2$

$= 0.9498$

By applying Milne's corrector formula again,

$y_{4,c} = y_2 + \frac{h}{3}(y'_2 + 4y'_3 + y'_4)$

$= 1.12 + \frac{0.1}{3}(0.6523 + 4(0.7979) + 0.9500)$

$= 1.2798$

$y_{4,c_1} = y_{4,c_2} = 1.2798$ is the require soln.

$\therefore y(0.4) = 1.2798$

4. Find $y(0.8)$ by milne's method for the equ $y' = y - x^2$ $y(0)=1$ Obtain the starting values by Taylor series method.

Given $y' = y - x^2 \rightarrow ①$

and $x_0 = 0$, $y_0 = 1$ and $h = 0.2$

First we find the starting value $y(0.2)$ $y(0.4)$ & $y(0.6)$ by Taylor's method

The taylor's Algorithm is

$y_1 = y_0 + \frac{h}{1!}y_0 + \frac{h^2}{2!}y''_0 + \frac{h^3}{3!}y'''_0 + \cdots$

Diff ① w.r.to 'x' we get

$y'' = y' - 2x$

$y''' = y'' - 2$

$\therefore y'_0 = (y')(x_0, y_0) = y_0 - x_0^2 = 1$

$\therefore y''_0 = (y'')(x_0, y_0) = y'_0 - 2x_0 = 1$

$y'''_0 = (y''')(x_0, y_0) = y''_0 - 2 = 1 - 2 = -1$ 27

Using this in ② we get

$y_1 = 1 + 0.2 + \frac{(0.2)^2}{2} + \frac{(0.2)^3}{6}(-1)$

is, $y(0.2) = 1.2187$

Now, $y_2 = y_1 + \frac{h}{1!} y'_1 + \frac{h^2}{2!} y''_1 + \frac{h^3}{3!} y'''_1 + \dots$ → ③

$x_1 = x_0 + h = 0.2$

$\therefore (y'_1) = (y')(x_1, y_1) = y_1 - x_1^2 = 1.2187 - (0.2)^2$
$= 1.1787$

$(y''_1) = (y'')(x_1, y_1) = y'_1 - 2x_1 = 1.1787 - (0.4)$
$= 0.7787$

$y'''_1 = (y''')(x_1, y_1) = y''_1 - 2 = -1.2213$

using these values in ③ we get

$y_2 = 1.2187 + (0.2)(1.1787) + \frac{(0.2)^2}{2}(0.7787) + \frac{(0.2)^3}{6}(-1.2213)$

ie) $y(0.4) = 1.4684$

Now, $y(0.6) = y_3 = y_2 + \frac{h}{1!} y'_2 + \frac{h^2}{2!} y''_2 + \frac{h^3}{3!} y'''_2 + \dots$ → ④

$y'_2 = (y')(x_2, y_2) = y_2 - x_2^2 = 1.4684 - 0.4^2 = 1.3084$

$y''_2 = (y'')(x_2, y_2) = y'_2 - 2x_2 = 1.3084 - 0.8 = 0.5084$

$y'''_2 = (y''')(x_2, y_2) = y''_2 - 2 = 0.5084 - 2 = -1.4916$

Using these values in ④ we get

$y_3 = 1.4684 + (0.2)(1.3084) + \frac{(0.2)^2}{2}$

$(0.5084) + \frac{(0.2)^3}{6}(-1.4916)$

$y(0.6) = 1.7383$

Milne's predictor formula is

$$y_{n+1,P} = y_{n-3} + \frac{4h}{3}[2y'_{n-2} - y'_{n-1} + 2y'_n]$$

put $n=3$ we get

$$y_{4,P} = y_0 + \frac{4h}{3}[2y'_1 - y'_2 + 2y'_3]$$

28

Now, $y'_3 = (y')(x_3, y_3) = y_3 - x_3^2$

$= 1.7383 - (0.6)^3$

$= 1.5223$

$\therefore y_{4,P} = 1 + \frac{4(0.2)}{3}(2(1.1787) - 1.3084 + 2(1.5283))$

$= 2.0916$

Milne's corrector formula is

$$y_{n+1,C} = y_{n-1} + \frac{h}{3}[y'_{n-1} + 4y'_n + y'_{n+1}]$$

put $n=3$, we get $y_{4,C} = y_2 + \frac{h}{3}(y'_2 + 4y'_3 + y'_4)$ $\to$ ⑥

Now, $y'_4 = (y')(x_4, y_4) = y_4 - x_4^2$

$= 2.0916 - x_4^2$

$= 2.0916 - 0.8^2$

$= 1.4516$

$y_{4,C} = 1.1484 + \frac{0.2}{3}[1.3084 + 4(1.5223) + 1.4576]$

$= 2.0583$

$\therefore y(0.8) = 2.0583$

The End